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Solutions to the problems of the Final Examination of Physical Systems April 10, 2017

Problem #1. Complex Numbers and Differential Equations (10 Points)

1. a) the norm of X is equal to $\exp(2)$. The norm of Y is equal to 1.

   b)

$$X = \exp(2) * (\cos(1) + i\sin(1))$$
$$X = 3.99 + 6.22i$$
$$Y = (\cos(5) + i\sin(5))$$
$$Y = 0.28 - 0.96i$$

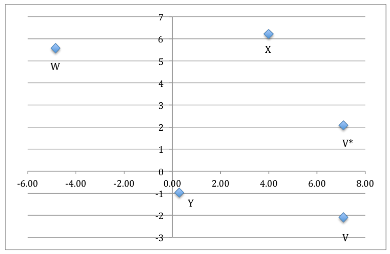
   c)

$$V = X * Y = (3.99 + 6.22i) * (0.28 - 0.96i)$$
$$V = 7.0884 - 2.09i$$
$$W = X/Y = \frac{(3.99 + 6.22i)}{(0.28 - 0.96i)}$$
$$W = \frac{(3.99 + 6.22i)}{(0.28 - 0.96i)} * \frac{(0.28 + 0.96i)}{(0.28 + 0.96i)}$$
$$W = \frac{(-4.85 + 5.57i)}{(1)} = -4.85 + 5.57i$$

   d)

$$V^* = 7.0884 + 2.09i$$

   e) See the plot.



2.

$$\lambda^2 + 8\lambda + 15 = 0$$

$$(\lambda + 3)(\lambda + 5) = 0$$

$$\lambda_1 = -3 \quad \lambda_2 = -5$$

$$x(t) = \alpha_1 \exp(-3t) + \alpha_2 \exp(-5t)$$

$$x(0) = \alpha_1 + \alpha_2 = 0$$

$$x'(0) = -3\alpha_1 - 5\alpha_2 = 8$$

$$\alpha_1 = 4$$

$$\alpha_2 = -4$$

general solution:

$$x(t) = 4\exp(-3t) - 4\exp(-5t)$$

Problem #2. Mechanical Oscillations without Damping (17 Points)

a)

$$m \cdot a = -k \cdot x$$
$$m\frac{d^2x}{dt^2} + k \cdot x = 0$$

b)

$$\begin{aligned} x(t) =& Asin(\omega t) \\ x'(t) =& \omega Acos(\omega t) \\ x''(t) =& -\omega^2 Asin(\omega t) \end{aligned}$$

Fill in for the differential equation to find

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

c) The acceleration is maximum at the position where the displacement is maximum. Therefore:

$$k = \frac{m \cdot a_{max}}{x_{max}} = 5 \quad \text{N/m}$$

d) Two springs in series:

$$\begin{aligned} \frac{1}{k_{total}} =& \frac{1}{k_1} + \frac{1}{k_2} \quad \text{where} \quad k_1 = 0.5 \cdot k_2 \quad \text{and} \quad k_{total} = 5 \quad \text{N/m} \\ \frac{1}{k_{total}} =& \frac{3}{k_2} \quad \text{so} \quad k_2 = 15 \quad \text{N/m} \quad \text{and} \quad k_1 = 7.5 \quad \text{N/m} \end{aligned}$$

e) Two springs in parallel (both sides of the block):

$$\begin{aligned} k_{total} =& k_1 + k_2 \\ k_1 =& 0.5 \cdot k_2 \\ k_{total} =& 1.5k_2 \\ k_2 =& 3.33 \quad \text{N/m} \\ k_1 =& 1.66 \quad \text{N/m} \end{aligned}$$

f)

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\begin{aligned}
x(t) =& Asin(\omega t) \\
x'(t) =& \omega Acos(\omega t) \\
x''(t) =& -\omega^2 Asin(\omega t) \\
\omega = \omega_0 =& \sqrt{\frac{k}{m}} = 3.16 \quad \text{rad/s} \\
v_{max} =& \omega \cdot A = 1.58 \quad \text{m/s} \\
a_{max} =& -\omega^2 \cdot A = 5 \quad \text{m/s}^2 \text{ as it should be.}
\end{aligned}
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Problem #3. Mechanical Oscillations with Damping (17 Points)

a) It is given that the appearent force has a magnitude of $mF(t)/M$. With the given numbers, one obtains $F_{apparent} = mF(t)/M = 10 \text{ kg} \cdot 5000 \text{ N}/1000 \text{ kg} = 50 \text{ N}$ Under this force, the spring will compress a distance of $x = F_{apparent}/k = 50 \text{ N}/1000 \text{ N/m} = 5$ cm. Hence, the poles should be at least 5 cm from the equilbrium position.

b) The forces on the ball are the apparent force, the spring force and friction. The sum of these forces is given by

$$\Sigma F = -kx - cv + \frac{m}{M}F(t)$$

From Newtons second law we then obtain:

$$ma = -kx - cv + \frac{m}{M}F(t)$$

or

$$ma + cv + kx = \frac{m}{M}F(t)$$

This gives us the following differential equation for the motion of the ball:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k \cdot x = \frac{m}{M} \cdot F(t)$$

c) Divide the differential equation by m:

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m} \cdot x = \frac{1}{M} \cdot F(t)$$

Then introduce that:

$$\omega_0^2 = \frac{k}{m} \qquad 2\gamma = \frac{c}{m}$$

To obtain:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 \cdot x = \frac{1}{M} \cdot F(t)$$

Then we know from the theory of undriven damped oscillations that $\gamma = \frac{c}{2m} = 100 \text{ Ns/m} \ / \ 20 \text{ kg} = 5 \text{ s}^{-1}$

We also know that a damped oscillator oscillates with a trancient frequency of $\omega^2 = \omega_0^2 - \gamma^2$.
Therefore, one can calculate that $\omega_0^2 = k/m = 1000$ N/m / 10 kg = 100 (rad/s)^2.
This gives: $\omega^2 = 100$ (rad/s)^2 – 25 (rad/s)^2 = 75 (rad/s)^2 Hence:
$\omega = 8.66$ rad/s

d) We have given that the bowling ball starts from its equlibrium position with a speed of 50 km/h. Hence, we get that $x(0) = 0$ and $x'(0) = 50$ km/h = 13.9 m/s. Filling in the solution gives:

$$x(t=0) = \left. Be^{-\gamma t}\cos(\omega t - \alpha)\right|_{t=0} = B\cos(\alpha) = 0$$

Since $B \neq 0$ we must have that $\cos(\alpha) = 0$ hence that $\alpha = \pi/2 + k\pi$. We select a value: $\alpha = \pi/2$.
Differentiating the solution gives:

$$x'(t=0) = \left. -\gamma Be^{-\gamma t}\cos(\omega t - \alpha) - \omega Be^{-\gamma t}\sin(\omega t - \alpha)\right|_{t=0} = -\gamma B\cos(\alpha) - \omega B\sin(\alpha)$$

With $\alpha = \pi/2 + k\pi$ this becomes:

$$x'(t=0) = -\gamma B \cdot 0 - \omega B \cdot 1 = -\omega B = 13.9 \text{ m/s}$$

From question 4 we know that $\omega = 8.66$ rad/s. This gives $B = -1.6$ m.

e) We calculate maxima by differentiating:

$$x'(t) = -\gamma Be^{-\gamma t}\cos(\omega t - \alpha) - \omega Be^{-\gamma t}\sin(\omega t - \alpha) = 0$$

$$-\gamma\cos(\omega t - \alpha) - \omega\sin(\omega t - \alpha) = 0$$
$$\tan(\omega t - \alpha) = \frac{\sin(\omega t - \alpha)}{\cos(\omega t - \alpha)} = -\frac{\gamma}{\omega} = -\frac{5 \text{ rad/s}}{8.66 \text{ rad/s}} = -0.577$$
$$\omega t - \alpha = \arctan(-0.577) + k\pi$$
$$\omega t = \pi/2 + k\pi + \arctan(-0.577) = \pi/2 - 0.524 + k\pi$$

The first maximum for $t > 0$ is at $k = 0$. Hence $\omega t = \pi/2 - 0.524$. Hence

$t = 0.1209$ s.
This gives a displacement of

$$x(t) = Be^{-\gamma t}\cos(\omega t - \alpha) = -1.6\,\text{m}\cdot e^{-5\ \text{s}^{-1}\cdot 0.1209}\cdot\cos(\pi/2 - 0.524 - \pi/2) = -0.757\,\text{m}$$

Now obviously 0.757 m displacement is much more than the 5 cm distance to the poles, hence the airbag will fire and Mr. Bean will survive the collision. Notice that the ball will start moving towards the poles, which is the negative direction. Hence, the first maximum is the appropriate one.

f) Obviously the airbag will activate if the amplitude of the solution exceeds the distance to the poles. call this distance d. Then we should solve:

$$\frac{F_0/M}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} > d$$

$$\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} < \frac{F_0}{Md}$$

$$(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 < \frac{F_0^2}{M^2d^2}$$

Define $x = \omega^2$

$$x^2 - 2\omega_0^2 x + \omega_0^4 + 4\gamma^2 x - \frac{F_0^2}{M^2d^2} < 0$$

$$1\cdot x^2 + \{4\gamma^2 - 2\omega_0^2\}\cdot x + \{\omega_0^4 - \frac{F_0^2}{M^2d^2}\} < 0$$

Use abc-formula with:

$$a = 1 \quad b = 4\gamma^2 - 2\omega_0^2 = -100 \quad c = \omega_0^4 - \frac{F_0^2}{M^2d^2} = 1900$$

$$D = b^2 - 4ac = 2400$$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$x = 74.5 \quad \text{or} \quad x = 25.5$$

since $a = 1$ we have a parabola with a minimum. since the inequality is $<$, this means it is satisfied between these values:

$$25.5 < x < 74.5$$

Since $x = \omega^2$ we obtain

$$5.05 \text{ rad/s} < \omega < 8.63 \text{ rad/s}$$

If ω is between the boundaries above, the airbag will fire.

g) If 2 speed bumps have a distance of 50 m and Mr. Bean is driving with a speed of 30 km/h then the time between two succeeding speed bumps is $t = s/v = 50 \text{ m}/30 \text{ km/h} = \text{m}/(30/3.6) \text{ m/s} = 6.0 \text{ s}$. Assuming this time to be the time of one oscillation, we obtain an angular frequency of $\omega = 2\pi/T = 2\pi/6.0 \text{ s} = 1.047 \text{ rad/s}$. Since this is outside of the boundaries specified in question 8, the airbag will NOT fire and Mr. Bean can drive safely.

Problem #4. Electric Circuit – I (18 Points)

a)

$$R_{equiv.} = \frac{(R_1 + R_2)(R_4 + R_x)}{R_1 + R_2 + R_3 + R_4}$$

b) A: $I_{in} - I_1 - I_3 = 0$
B: $-I_{in} + I_2 + I_x = 0$

c) $V - I_1(R_1 + R_2) = 0 \Rightarrow I_1 = \frac{V}{R_1+R_2}$

$V - I_3(R_3 + R_x) = 0 \Rightarrow I_3 = \frac{V}{R_3+R_x}$

The voltage drop over R_2 (between C and B, namely V_1) and R_x (between D and B, namely V_2) are then:

$V_1 = R_2 \frac{V}{R_1+R_2} = V \frac{R_2}{R_1+R_2}$ $\quad$ $V_2 = R_x \frac{V}{R_3+R_x} = V \frac{R_x}{R_3+R_x}$

d) $V_m = V_1 - V_2 = V(\frac{R_2}{R_1+R_2} - \frac{R_x}{R_3+R_x})$

e) $R_x = R_3 \frac{\frac{R_2}{R_1+R_2} - \frac{V_m}{V}}{1-(\frac{R_2}{R_1+R_2} - \frac{V_m}{V})} = 10 \frac{\frac{5}{6}-0.1}{1-(\frac{5}{6}-0.1)} = 27.5 \quad \text{k}\Omega$

f) Take the two limits when $R_x = 0$ and $R_x \to \infty$:

$R_x = 0 \Rightarrow V_m = 10(\frac{5}{6}) = 8.33 \quad \text{V}$
$R_x \to \infty \Rightarrow V_m = 10(\frac{5}{6} - 1) = -1.66 \quad \text{V}$

g) Set $V_m = 0$ and obtain $\frac{R_1}{R_2} = \frac{R_3}{R_x}$

Problem #5. Electric Circuit – II (18 Points)

a) $Z_C = \frac{-i}{\omega C}$ or $\frac{1}{i\omega C}$

b) $A_C = \frac{V_c}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{1 - i\omega RC}{1+\omega^2 R^2 C^2}$ This is Eq. V.23 of the syllabus.

c) $|A_C| = \sqrt{Re(A_C)^2 + Im(A_C)^2} = \frac{1}{\sqrt{1+(\omega RC)^2}}$ This is Eq. V.24 of the syllabus.

$\phi_{A_C} = -\arctan(\omega RC)$ This is Eq. V.25 of the syllabus.

d) This is a low pass filter because:
$\lim_{\omega \to 0} |A_C| = 1$
$\lim_{\omega \to \infty} |A_C| = 0$

e) $Z_L = i\omega L$

f) $\omega = 100\pi$
$\phi = \arctan \frac{R}{\omega L - \frac{1}{\omega C}}$ This is obtained from Eq. V.47 of the syllabus.
For $\phi = \pi/4$, we have:
$R = \omega L - \frac{1}{\omega C} \Rightarrow C = \frac{1}{\omega^2 L - \omega R}$
With the parameters of the problem, $C = 57.5\ \mu$F

g) $\omega_{res} = \sqrt{\frac{1}{LC} - \frac{1}{2}\frac{R^2}{L}} \approx \sqrt{\frac{1}{LC}} = 310.8$ Hz

Problem #6. Business Dynamics (20 Points)

a) Machine E (in station 4)
$r_b = \frac{60\text{min}}{30\text{min/unit}} = 2$ units/hr

b) $W_0 = r_b * T_0$
$= 2 * 2 = 4$ units
(Note that $T_0 = 20 + 50 + 20 + 30 = 120$ min)

c) The new throughput is $TH = \frac{60\text{min}}{50\text{min/unit}} = 1.2$ units/hr
In this case, the bottleneck is shifting from E to B. The throughput was 2 units/hr (see part a)). So, yes. It makes a difference.

d) First product is finished after 120 min. After that it produces at the bottleneck rate.
First 2 hours: 1 product
Last 6 hours: 6*1.2 = 7.2 units
Total: 8.2 products
So, 8 complete products.

e) Forecasting Fluctuating demand Order batching
Price variability Promotions Large orders in case of inventory shortage
Small order in case of large inventory
1 point for each correct cause + explanation.

f) The right-hand side represents the driving force in business dynamics. This is generated by the next-stage demand (customer demand).

g) The damping factor is given by $\frac{\beta+\epsilon}{2T}$.

h) Using the conditions $\gamma < \omega_0$ and requiring that one has a bound solution, one obtains: $0 < (\beta + \epsilon) < 2\sqrt{\frac{T}{\tau}}$