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Solutions to the problems of the Re-Examination of Physical Systems June 30, 2017

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Problem #1. Complex Numbers and Differential Equations (10 Points)

Part 1)

a. i) $R = 5 + 5i$

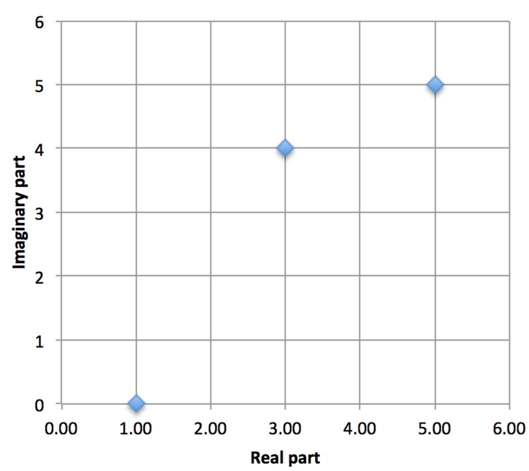
ii) modulus $= \sqrt{5^2 + 5^2} = \sqrt{50}$ phase $= \arctan \frac{5}{5} = 0.785$

b. i) $S = 3 + 4i$

ii) modulus $= \sqrt{3^2 + 4^2} = 5$ phase $= \arctan \frac{4}{3} = 0.927$

c. i) $T = 1$

ii) modulus $= 1$ phase $= \arctan \frac{0}{1} = 0$



Part 2)

$$\lambda^2 = -25$$
$$\lambda = 5i$$
$$y(t) = C\exp(5i * t) = C_1 * \cos(5t) + C_2 \sin(5t)$$
$$y(0) = C_1 = 2$$
$$y'(t) = -5C_1 \sin(5t) + 5C_2 \cos(5t)$$
$$y'(0) = 5C_2 = 5$$
$$C_2 = 1$$

General solution:

$$y(t) = 2\cos(5t) + \sin(5t)$$

Problem #2. Mechanical Oscillations without Damping (17 Points)

a) Under the assumption that θ is very small:

$$\sin(\theta) \approx \theta$$

$$\omega_0^2 = \frac{g}{L}$$

b)

$$\begin{aligned}
\theta =& \theta_0 \sin(\omega_0 t + \alpha) \\
\theta' =& \omega_0 \theta_0 \cos(\omega_0 t + \alpha) \\
\theta'' =& -\omega_0^2 \theta_0 \sin(\omega_0 t + \alpha) = -\omega_0^2 \theta
\end{aligned}$$

Fill in for the differential equation to find that it is indeed the solution to the differential equation.

c)

$$\begin{aligned}
\omega_0 = \sqrt{\frac{g}{L}} &= 1.98\,\mathrm{rad/s} \\
T = \frac{2\pi}{\omega} &= 3.17\,\mathrm{s}
\end{aligned}$$

d)

$$\begin{aligned}
h = L - L\cos(\theta) &= 2.5 - 2.5\cos(20^o) = 0.15\,\mathrm{m} \\
E_{pot} = mgh &= 7850 \cdot 9.81 \cdot 0.15 = 11610\,\mathrm{J} = 11.6\,\mathrm{kJ}
\end{aligned}$$

e) v_{max} occurs at the equilibrium point $h = 0$ or $\theta = 0°$,
a_{max} occurs at maximum amplitude $\theta = 20°$

$$\begin{aligned}
E_{\text{Tot.}} = E_{\text{kin}} + E_{\text{pot}} &= \frac{1}{2}mv^2 + mgh \\
E_{\text{Tot.}} &= 11.6\,kJ
\end{aligned}$$

f) Nothing, the period, the speed and the acceleration are independent of the mass.

Problem #3. Mechanical Oscillations with Damping (17 Points)

a) Assume the water has zero gravitational potential energy. Assume that Dagoberts closest position is height h above the water (h can be negative). Define $h_0 = 200m$. Hence, Dagoberts total energy before his free fall equals $E = mgh_0$. At the closest point to the water, Dagoberts speed is zero (oscillation extremum), hence his energy is a sum of grativational potential energy $E_g = mgh$ and spring potential energy $E_s = \frac{1}{2}ku^2$ where u is the distance over which the rope is strechted. Hence we now have to define u in terms of h. Since we know that the total lenght of the rope has to equal $h_0 - h$ and the unstrechted length of the rope is $l = 50$ m, this would be $u = h_0 - h - l$. Hence balancing the total energy gives:

$$mgh_0 = mgh + \frac{1}{2}k(h_0 - h - l)^2$$

1 point
Define $x = h_0 - h$. Then we can rewrite our equation as:

$$mgx = \frac{1}{2}k(x-l)^2 = \frac{1}{2}kx^2 - kxl + \frac{1}{2}kl^2$$

Or we can write it as:

$$\frac{1}{2}kx^2 - (kl + mg)x + \frac{1}{2}kl^2 = 0$$

Solving this with the abc formula gives:

$$a = \frac{1}{2}k = 4 \text{ J/m}^2 \qquad b = -(kl + mg) = -792.40 \text{ J/m} \qquad c = \frac{1}{2}kl^2 = 10000 \text{ J}$$

$$D = b^2 - 4ac = 4.6790 \cdot 10^5 \text{ J}^2/\text{m}^2$$

$$x = 13.546m \quad \text{or} \quad x = 184.55 \text{ m}$$

1 point
The first solution gives a height h above $h_0 - l$ which means the rope is not even stretched. Hence this answer is unphysical. The second solution gives a height $h = 15.45$ m above the water. Hence, Dagobert does NOT get wet. 1 point

b) After the oscillation damped out, this is just a matter of balancing gravity with the spring force.

$$mg = k * u$$

This gives a stretch in the rope of $u = 49.050$ m. Hence, the equilibrium above the water is at a height $h = h_0 - l - u = 100.95$ m. 1 point

c) Dabogerts natural oscillation frequency equals $\omega_0 = \sqrt{k/m} = 0.45$ rad/s. The time between Dagoberts closest point to the water and his highest point immediately afterwards is half the oscillation period. Hence, this time equals

$$t = \frac{T}{2} = \frac{\pi}{\omega_0} = 7.0248 \text{ s}$$

1 point

The amplitude close to the water equals

$$A_0 = 100.95 \text{ m} - 15.45 \text{ m} = 85.5 \text{ m}$$

The amplitude of the first swing upwards Reaches a height of $200 \text{ m} - 20\text{m} = 180$ m and therefore equals

$$A_1 = 180 \text{ m} - 100.95 \text{ m} = 79.05 \text{ m}$$

1 point

We know that the amplitude decays exponentially, hence we can solve gamma from the following equation:

$$\frac{A_1}{A_0} = e^{-\gamma t}$$

0.5 point

Where we calculated t before. This results in $\gamma = 0.011$ 1/s. 0.5 point

With the assumption of 100 m equilibrium point and 10 m above the water, one would get $A_0 = 90$ m and $A_1 = 80$ m. This results in a damping factor of $\gamma = 0.017$ 1/s.

d) Our damping factor would result in an oscillation frequency of $\omega = \sqrt{{\omega_0}^2 - \gamma^2} = 0.44707$ rad/s while $\omega_0 = 0.44721$ rad/s. Hence, our assumption that Dagobert oscillates almost with the natural frequency was justified. 1 point

e) An undriven damped harmonic oscillation is described by

$$x(t) = Be^{-\gamma t}\cos(\omega t - \alpha)$$

1 point

This will give a speed of

$$x'(t) = -\gamma Be^{-\gamma t}\cos(\omega t - \alpha) - \omega Be^{-\gamma t}\sin(\omega t - \alpha)$$

1 point

We know that the maximum speed occurs at the equilibrium position. (since $\omega \approx \omega_0$ this is also true for damped oscillations). Hence we can compute B and α and then solve t at the equilibrium position. Then we can compute the speed. Define your starting time $t = 0$ at the moment when Dagobert is at his lowest point. As calculated in part c), this means that we start from rest (speed is zero at the maximum position) at an amplitude of $A_0 = 85.5$ m. This means that $\alpha = 0$ and $B = A_0 = 85.5$ m. 1 point. Solving for $x(t) = 0$ gives then

$$A_0 e^{-\gamma t}\cos(\omega t) = 0$$
$$\cos(\omega t) = 0$$
$$\omega t = \frac{\pi}{2} + k \cdot \pi$$

We wanted the maximum speed Dagobert will ever have. Since the oscillation is damped, this happens at the first time when Dagobert crosses the equilibrium, hence at $k = -1$, when he makes his first free fall. Hence, $t = -\frac{\pi}{2\omega}$. 1 point. Now fill this in into our equation for speed:

$$x'(t = -\pi/(2\omega)) = -\gamma Be^{+\frac{\gamma\pi}{2\omega}}\cos(-\pi/2) - \omega Be^{+\frac{\gamma\pi}{2\omega}}\sin(-\pi/2) = +\omega Be^{+\frac{\gamma\pi}{2\omega}} = +39.75\,\mathrm{m/s}$$

The plus sign is due to the fact that we define the positive displacement in the downward direction, while Dagobert also crosses the equilibrium (during his first free fall at time $t < 0$) in the downward direction. Hence, $v = 39.75$ m/s. 1 point

With $\gamma = 0.017$ 1/s and $A_0 = 90$ m one obtains: $v = 42.663$ m/s.

f) If Dagobert has to hit the water due to the steady state motion, The steady state amplitude has to be bigger than the equilibrium height of 100.95 m. Call this equilibrium height h_{eq}. Then we need to solve

$$A \geq h_{eq}$$

1 point

Solving the equality first gives us:

$$A = h_{eq}$$
$$\frac{F_0/m}{\sqrt{(\omega^2 - {\omega_0}^2)^2 + 4\gamma^2\omega^2}} = h_{eq}$$
$$\frac{1}{\sqrt{(\omega^2 - {\omega_0}^2)^2 + 4\gamma^2\omega^2}} = \frac{mh_{eq}}{F_0}$$
$$(\omega^2 - {\omega_0}^2)^2 + 4\gamma^2\omega^2 = \frac{{F_0}^2}{m^2{h_{eq}}^2}$$

Introduce $x = \omega^2$ and rewrite the equation to

$$(x - {\omega_0}^2)^2 + 4\gamma^2 x - \frac{{F_0}^2}{m^2{h_{eq}}^2} = 0$$

Which will reduce

$$x^2 + (4\gamma^2 - 2{\omega_0}^2)x + ({\omega_0}^4 - \frac{{F_0}^2}{m^2{h_{eq}}^2}) = 0$$

1 point

Solving this with the abc-formula gives

$$a = 1 \qquad b = 4\gamma^2 - 2{\omega_0}^2 = -0.39950 \qquad c = {\omega_0}^4 - \frac{{F_0}^2}{m^2{h_{eq}}^2} = 0.039387$$

This gives

$$D = b^2 - 4 * a * c = 0.0020544$$

Which returns

$$x = 0.22241 \quad \text{or} \quad x = 0.17709$$

This gives us two frequencies of ω:

$$\omega = 0.42082 \quad \text{or} \quad \omega = 0.47161$$

1 point

Knowning that the resonance frequency is in between (as is the natural frequency), any frequency obeying $0.42082 \leq \omega \leq 0.47161$ will cause that Dagobert hits the water. Hence the smallest dirving frequency Donald has to use is $\omega = 0.42082$ rad/s. 1 point

In case an equilibrium height of 100 m is used, a minimum frequency of $\omega = 0.42051$ rad/s is obtained.

Problem #4. Electric Circuit – I (18 Points)

a) $\Sigma V = 0$ around a loop and $\Sigma I = 0$ at any node.

b) Current Law:

$$I_1 - I_2 - I_3 - I_4 - I_5 = 0 \quad (1)$$
$$I_4 + I_5 + I_6 - I_7 = 0 \quad (2)$$
$$I_3 + I_7 - I_{\text{in}} = 0 \quad (3)$$

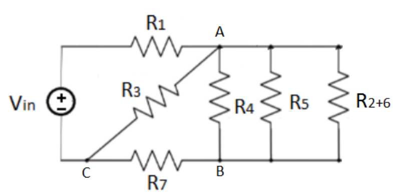
c) Voltage Law:

$$V_{\text{in}} - V_1 - V_3 = 0 \quad (4)$$
$$V_3 - V_4 - V_7 = 0 \quad (5)$$
$$V_4 - V_5 = 0 \quad (6)$$
$$V_5 - V_2 - V_6 = 0 \quad (7)$$

d) The first step is to realize that R_2 and R_6 are connected in series, thus they can be added:

$$R_{2+6} = R_2 + R_6 = 3\,\Omega + 7\,\Omega = 10\,\Omega$$

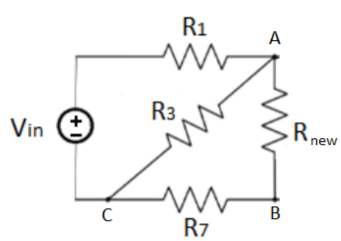
The circuit can then be redrawn:



The R_4, R_5, and R_{2+6} resistors were drawn this way to make it clear that they are all in parallel. Because parallel resistors add inversely, their equivalent is:

$$R_{\text{new}} = [1/R_4 + 1/R_5 + 1/R_{2+6}]^{-1} = [1/20 + 1/10 + 1/10]^{-1} = [5/20]^{-1} = 4\,\Omega$$

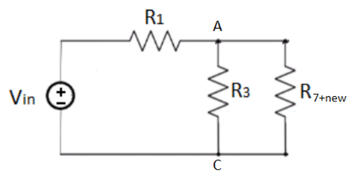
Again, we redraw the diagram:



Since R_7 and R_{new} are connected in series, we add them:

$$R_{7+new} = R_7 + R_{\text{new}} = 6\,\Omega + 4\,\Omega = 10\,\Omega$$

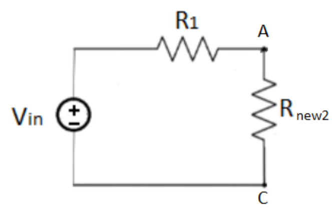
Redrawing the diagram:



We now find the value of the remaining parallel circuit:

$$R_{\text{new2}} = [1/R_3 + 1/R_{7+new}]^{-1} = [1/10 + 1/10]^{-1} = [2/10]^{-1} = 5\,\Omega$$

Redrawing one last time:



The equivalent resistance is, therefore, the series addition between R_1 and R_{new2}:

$$R_{\text{equiv.}} = R_1 + R_{\text{new2}} = 5\,\Omega + 5\,\Omega = 10\,\Omega$$

e)

$$\begin{aligned} I_1 &= 1\,\text{A}; \quad I_2 = 0.2\,\text{A}; \quad I_3 = 0.5\,\text{A}; \\ I_4 &= 0.1\,\text{A}; \quad I_5 = 0.2\,\text{A}; \quad I_6 = 0.2\,\text{A}; \\ I_7 &= 0.5\,\text{A} \end{aligned}$$

f)

$$\begin{aligned} V_1 &= 5\,\text{V}; \quad V_2 = 0.6\,\text{V}; \quad V_3 = 5\,\text{V} \\ V_4 &= 2\,\text{V}; \quad V_5 = 2\,\text{V}; \quad V_6 = 1.4\,\text{V} \\ V_7 &= 3\,\text{V} \end{aligned}$$

g) All voltages and currents from e) and f) times one third.

Problem #5. Electric Circuit – II (18 Points)

a)

$$Z_{\text{tot}} = Z_L + Z_R = i\omega L + R$$

b)

$$A_L = \frac{V_L}{V_{\text{in}}} = \frac{i\omega L}{R + i\omega L} = \frac{\omega^2 L^2 + i\omega LR}{R^2 + \omega^2 L^2}$$

This is Equation (V.32) of the Syllabus.

c)

$$|A_L| = \sqrt{Re(A_L)^2 + Im(A_L)^2} = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi_{A_L} = \arctan(\frac{Im(A_L)}{Re(A_L)}) = \arctan(\frac{R}{\omega L})$$

These are Equations (V.34) and (V.35) of the Syllabus.

d) $|A_R| = \frac{R}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}$ Equation (V.49) of the Syllabus.

e) $\omega = 2\pi \times 50$ rad/s

$C = 18.4\ \mu$F

Problem #6. Business Dynamics (20 Points)

a) RED: Machine C
BLUE: Machine D

b) $\#\text{RED} = r_{\text{b,RED}} = \frac{60\,\text{min}}{5\,min/unit} = 12\,units/\text{h}$
$\#\text{BLUE} = r_{\text{b,BLUE}} = \frac{60\,\text{min}}{6\,min/unit} = 10\,units/\text{h}$

c) Add an extra machine D; and
Reduce the process time of machine D.

d) Each 5 minutes (bottleneck) machine E produces a RED product.
So utilization is 3/5=60%.

e) Machine A takes 4 minutes to produce a RED and a BLUE product. Therefore, station C and D remain the bottleneck. Machine F has an input of 10 units/hour of BLUE and 12 units/hour of RED. (see b)) Total input for machine F is 22 units/hour. Machine F produces a RED and a BLUE product at a rate of 15 units/hour. This will be done in alternating order.
So 7.5 RED products per hour
and 7.5 BLUE products per hour.

f) The bullwhip effect. It refers to a trend of larger and larger swings in inventory in response to changes in demand, as one looks at companies further back in the supply chain of a product. The following words or synonyms should be used correctly to obtain full points:
swings (oscillations), supply chain, demand changes.

g) Causes:
forecasting / fluctuating demand / order batching / price variability / large orders in case of inventory shortage / small order in case of large inventory / ...
Preventive measures: Steady prices / no order batching / stabilizing demand / stop (multiple) forecasting) / information sharing / ... (1 point for correct measure + explanation)

h) Eigen frequencey, $\omega_0 = \sqrt{\frac{1}{T\tau}}$.

i) Using the conditions $\gamma^2 > \omega_0^2$ and requiring that one has a bound solution, one obtains: $(\beta + \epsilon) > 2\sqrt{\frac{T}{\tau}} > 0$.