

Final Exam for Calculus 2 for IEM

Block 2A, 2022–2023



university of
groningen

INSTRUCTIONS TO CANDIDATES

1. The time allowed for this exam is 2 hours.
2. Attempt all 6 questions in this test. The total number of points available is 100. You will receive 10 points for free and your score will be your point total divided by 10.
3. The number of points you can get for each question is shown next to it.
4. In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.
5. Simple pocket calculators are allowed at this exam. Other electronic devices such as graphical/programmable calculators, tablets, laptops and mobile phones are not.
6. Books, notes and formula sheets are not allowed.
7. Please write your solutions on the accompanying pages and make sure that all pages have your name and student ID on them.

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM.

1 Let $f(x, y) = \frac{x^2y - x^3y^2}{x^2 + y^2}$.

a) Determine the limit $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$, or explain why it does not exist. [6 points]

b) Let $a(t) = 1 - \cos t$, $b(t) = \sin t$. Determine the limit

$$\lim_{t \rightarrow 0} \left(a(t) + 11, b(t) + 4, 2023(1 - f(a(t), b(t))) \right),$$

or explain why it does not exist. [6 points]

2 Let $D_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9, \quad z \geq 0\}$
and $D_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 4z^2 = 9, \quad z \geq 0\}$.

a) Find the volume enclosed by D_1 and D_2 . [6 points]

- b) Find the surface area of D_1 . **[6 points]**
- c) Let $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ constitute a continuous, differentiable vector field. Calculate

$$\iint_{D_2} \nabla \times \mathbf{F} \cdot d\mathbf{S} - \iint_{D_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}. \quad \mathbf{[6 points]}$$

3 Consider the surface given by $xy^3z + e^{yz+x} + \sin yz = 31$.

- a) Find a vector that is perpendicular to the surface at the point $(\ln 31, 0, 1)$. **[6 points]**
- b) Find an equation of the plane tangent to the surface at the point $(\ln 31, 0, 1)$. **[6 points]**
- c) If z can be expressed as a function of x and y , find $\frac{\partial z}{\partial y}$. **[6 points]**

4 Let \mathbf{a} expressed in spherical coordinates ρ, θ, ϕ be represented by $\rho = \sqrt{3}, \theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$. Let \mathbf{b} expressed in cylindrical coordinates r, θ and z , be represented by $r = \sqrt{2}, \theta = \frac{\pi}{4}, z = -\sqrt{2}$. Find the distance between \mathbf{a} and \mathbf{b} . **[6 points]**

5 The position of a particle at time $t \in [\pi, 3\pi]$ is given by

$$\mathbf{r}(t) = \sin t \mathbf{i} + \sqrt{2} \cos t \mathbf{j} + \sin t \mathbf{k}.$$

- a) Calculate the length of the path traversed by the particle from $t = \pi$ to $t = 3\pi$. **[6 points]**
- b) Calculate the curvature of the particle's path at $t = 2\pi$. **[6 points]**

6 Let $G(x, y) = x^2 + x^2y^2 + y^2$.

- a) Find all stationary points of G . **[6 points]**
- b) Find the directional derivative $D_{\mathbf{u}}G(2, -2)$ for any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle \in \mathbb{R}^2$. **[6 points]**
- c) Check for each stationary point in part a) whether it corresponds to a local maximum, a local minimum, or a saddle point. **[6 points]**
- d) Use Lagrange multipliers to find the maximum and minimum of $G(x, y)$ for the case that $x^2 + 2y^2 = 1$. **[6 points]**