Mock Exam of the course:
Optimization in Engineering Systems

March 2023

The response should be explicit as indicated in the exam instructions

Question 1

Part (i): Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x_1, x_2) = \frac{1}{16}x_1^2 + 4x_2^2 + x_1 g(x_2)$$

where the scalar function $g(x_2)$ satisfies

$$\frac{\partial^2 g}{\partial x_2^2}(x_2) = 0, \quad \forall x_2 \in \mathbb{R}.$$

Under what additional condition on g, the function h is strictly convex?

Next, compute the minimum of h for the case $g(x) = \frac{1}{2}(x + 6)$.

Part (ii): Let A be a full row-rank $m \times n$ matrix with $m < n$, and $b \in \mathbb{R}^n$. Then, the linear equation $Ax = b$ has infinitely many solutions. Among all these solutions, we would like to find the one with minimum norm. This gives rise to the minimization problem

$$\begin{aligned} \text{Minimize}_{x \in \mathbb{R}^n} \quad & \frac{1}{2}x^\top x \\ s.t. \quad & Ax = b. \end{aligned}$$

Compute the optimal value $\frac{1}{2}(x^*)^T x^*$ by maximizing the *dual problem.* **Hint:** The matrix $AA^\top$ is invertible.

Part (iii): Consider the function

$$f(x_1, x_2, x_3) = (x_1 - a)^2 + x_2^2 + x_1(x_2 - a) + (x_3 - 2)^2 - \gamma \ln(x_3 + 1)$$

where a is the last digit of your student number, $\gamma \geq 0$, and $\ln(\cdot)$ denotes the natural logarithm. Write a gradient descent algorithm whose solutions converge to the minimizer of f.

Part (iv): Add the linear constraints

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_2 + 4x_3 &= 6 \end{aligned}$$

to the problem in Part (iii). Construct a primal-dual algorithm whose solutions converge to the minimizer of f under the above constraints.

Question 2 Let the cost function of a prosumer be given by

$$J(x) = \frac{1}{2}q_1 x^2 + c_1 x$$

and its utility function be given by

$$U(x) = q_2 \ln(x + c_2)$$

where c_1, c_2, q_1, q_2 are all positive constant, and x is a scalar decision variable. The aim is to minimize the *net* cost $F(x) = J(x) - U(x)$. Suppose you have two processors to solve this optimization problem .

Part (i): Formulate this as a distributed optimization problem, where processor 1 uses the cost parameters c_1 and q_1, and processor 2 uses parameters c_2 and q_2.

Part (ii): By using the Laplacian matrix, write down a primal-dual algorithm to solve the formulated distributed optimization problem.

Part (iii): Set $q_1 = 1$, and $q_2 = 16$. Take both $c_1 = c_2 = c$. Compute algebraically the optimal point x^* minimizing $F(x)$.

Question 3 Suppose we modify a resource allocation problem as

$$\begin{aligned} \text{Minimize}_{p_1,p_2,p_3,p_4} \quad & f_1(p_1) + f_2(p_2) + f_3(p_3) + (p_4 - 2)^2 \\ \text{subject to} \quad & \\ & p_1 + p_2 + p_3 + p_4 = d_1 + d_2 + d_3 + d_4 \end{aligned}$$

with $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 4$, where the blue terms indicate the modifications. How the distributed algorithm for the original problem should be be revised to provide a distributed solution to this modified resource allocation problem with 4 nodes? Note: (i) It is assumed a Laplacian matrix is used in both cases. (ii) You should not compute the whole revised algorithm, and instead should argue which terms should be added to the original algorithm as a result of the addition of the blue terms.