Problem 1: Convex analysis (10 points)

Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := \max\{3x - 6, -x + 2\}.$$

Calculate the subdifferential (set of subgradients) of f at the point $x = 2$, that is, $\partial f(2)$. Justify your answer or give the full derivation. Hint[1]

[1] To elaborate more on the max of two functions, at a particular x the value that the function $f(x) = \max\{3x - 6, -x + 2\}$ takes is the maximum of the two values: $f_1(x)$ and $f_2(x)$, where $f_1(x) = 3x - 6$ and $f_2(x) = -x + 2$. As an example, the absolute value function $g(x) = |x|$ is the max of two functions $g_1(x) = x$ and $g_2(x) = -x$. It might be helpful to plot the function f and understand how it behaves.

Important instructions regarding variables in your exam: Throughout the exam you will see some variables for which values have not been assigned. These are $\texttt{sd}_1$ and $\texttt{sd}_2$. These variables depend on your student ID. The variable $\texttt{sd}_1$ is the second last digit in your student ID. The variable $\texttt{sd}_2$ is the last digit in your student ID. For example, if your student ID is: s1234567, then $\texttt{sd}_1 = 6$ and $\texttt{sd}_2 = 7$.

Problem 2: Linear programming (10 points)

Find the solution to the following linear program **[8 points]**

$$\begin{aligned} \min \quad & -(\texttt{sd}_1 + 1)x_1 - (\texttt{sd}_2 + 2)x_2 \\ \text{subject to} \quad & |x_1| + |x_2| \leq 3, \end{aligned}$$

where $|x_1|$ is the absolute value of x_1 and $|x_2|$ is the absolute value of x_2. Again recall that you need to substitute $\texttt{sd}_1$ and $\texttt{sd}_2$ with the digits in your student ID (see the above instructions). Solve the problem using the graphical method, that is, plotting the feasibility set and then inferring the optimizer (do not use any software). Hint[1]

Does the solution change when you add another constraint to the above problem: $x_1 + x_2 \leq 1$ (justify your answer)? **[2 points]**

[1] The constraint $|x_1| + |x_2| \leq 3$ takes different forms based on if x_1 and x_2 are positive or negative. Hence, for each of the four quadrants of the x_1, x_2 plane, write down this constraint without the absolute values and then proceed to solve the linear program.

Problem 3: Properties of two-stage problems (20 points)

Consider the second-stage problem

$$
\begin{aligned}
\min \quad & 2y_1 + 3y_2 \\
\text{subject to} \quad & y_1 + y_2 \le 2x_1 - \xi x_2, \\
& y_1 \ge x_1, \\
& y_2 \le x_2, \\
& y_1 \ge 0, y_2 \ge 0.
\end{aligned}
$$

Do the following:

1. For any ξ (that is a scalar), compute $K_2(\xi)$ (see the footnote[1]). **[7 points]**
2. Let ξ take the following values with equal probability

   $$\{1, 2, 4\}.$$

   Then, compute K_2 (see the footnote[2]). **[7 points]**
3. Assume that ξ follows an exponential distribution with mean value equal to one. Then, compute K_2 (see the footnote[3]). **[6 points]**

[1] Recall that this set is the set of first-stage decisions that ensure feasibility in the second stage for the particular uncertainty ξ.

[2] Recall that this is the set of first-stage decisions that ensure feasibility in the second stage for all realizations of the uncertainty ξ.

[3] The probability density function of an exponential distribution with mean equal to one is:

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Problem 4: Computing EVPI and VSS (20 points)

Consider the following two-stage problem

$$\begin{aligned} \min \quad & x + 5\mathbb{E}_\xi[|x-\xi|] \\ \text{subject to} \quad & 0 \le x \le 10. \end{aligned}$$

Assume that the random variable ξ takes two values $\xi_1 = 0$ and $\xi_2 = 1$. Let the probability distribution be given as

$$p_1 = \mathbb{P}[\xi = \xi_1] = 0.5, \text{ and } p_2 = \mathbb{P}[\xi = \xi_2] = 0.5.$$

Do the following

1. Determine the value of the wait-and-see solution (WS). **[6 points]**
2. Determine the value of the recourse problem (RP). **[6 points]**
3. Determine the expected result of using the expected value solution (EEV). **[6 points]**
4. Using the above values, determine the expected value of perfect information (EVPI) and the value of stochastic solution (VSS). **[2 points]**

[Note: Do all calculations without the help of any electronic aid. Refer to the way we solved similar problems in the tutorial/lecture.]

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Problem 1: Convex analysis (10 points)

Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := |(\mathtt{sd}_1 + 1)x - (\mathtt{sd}_2 + 2)|$$

where you need to substitute $\mathtt{sd}_1$ and $\mathtt{sd}_2$ with the digits in your student ID (see the above instructions). Calculate the subdifferential (set of subgradients) of f at the point $\frac{\mathtt{sd}_2+2}{\mathtt{sd}_1+1}$, that is, $\partial f\left(\frac{\mathtt{sd}_2+2}{\mathtt{sd}_1+1}\right)$. Justify your answer or give the full derivation.

Problem 3: Properties of two-stage problems (20 points)

Consider the second-stage problem

$$\begin{aligned} \min \quad & y_1 + y_2 \\ \text{subject to} \quad & y_1 \leq x_1 - \xi_1 x_2, \\ & y_1 + y_2 \leq x_1 - \xi_2, \\ & y_1 \leq x_1, \\ & y_2 \leq x_2, \\ & y_1 \geq 0, y_2 \geq 0. \end{aligned}$$

Do the following:

1. For any $\xi = (\xi_1, \xi_2)$, compute $K_2(\xi)$ (see the footnote[1]). **[7 points]**
2. Let $\xi = (\xi_1, \xi_2)$ take the following values with equal probability

$$\{(1,1), (2,2), (4,2)\}.$$

   Then, compute K_2 (see the footnote[2]). **[7 points]**
3. Let $\xi = (\xi_1, \xi_2)$ take values $(u, 2)$ where u follows an exponential distribution with mean value equal to one. Then, compute K_2 (see the footnote[3]). **[6 points]**

[1]Recall that this set is the set of first-stage decisions that ensure feasibility in the second stage for the particular uncertainty ξ.

[2]Recall that this is the set of first-stage decisions that ensure feasibility in the second stage for all realizations of the uncertainty ξ.

[3]The probability density function of an exponential distribution with mean equal to one is:

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$