Important instructions regarding variables in your exam: Throughout the exam you will see some variables for which values have not been assigned. These are $\texttt{sd}_1$ and $\texttt{sd}_2$. These variables depend on your student ID. The variable $\texttt{sd}_1$ is the second last digit in your student ID. The variable $\texttt{sd}_2$ is the last digit in your student ID. For example, if your student ID is: s1234567, then $\texttt{sd}_1 = 6$ and $\texttt{sd}_2 = 7$.

Problem 1: Convex analysis (10 points)

Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := |(\texttt{sd}_1 + 1)x - (\texttt{sd}_2 + 2)|$$

where you need to substitute $\texttt{sd}_1$ and $\texttt{sd}_2$ with the digits in your student ID (see the first page of instructions). Calculate the subdifferntial (set of subgradients) of f at the point $\frac{\texttt{sd}_2+2}{\texttt{sd}_1+1}$, that is, $\partial f\left(\frac{\texttt{sd}_2+2}{\texttt{sd}_1+1}\right)$. Justify your answer or give the full derivation.

Problem 2: Linear programming (10 points)

Find the solution to the following linear program **[8 points]**

$$\begin{aligned} \min \quad & x_1 - (2 + \mathtt{sd}_1) x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 3, \\ & -(\mathtt{sd}_2 + 3) \leq x_1 \leq (\mathtt{sd}_2 + 1), \\ & -(\mathtt{sd}_2 + 1) \leq x_2 \leq (\mathtt{sd}_2 + 2). \end{aligned}$$

Again recall that you need to substitute $\mathtt{sd}_1$ and $\mathtt{sd}_2$ with the digits in your student ID (see the first page of instructions). Does the solution change when you add another constraint to the above problem: $x_1 + x_2 \leq 1$ (justify your answer)? **[2 points]**

Problem 3: Properties of two-stage problems (20 points)

Consider the second-stage problem

$$\begin{aligned} \min \quad & y_1 + y_2 \\ \text{subject to} \quad & y_1 \le x_1 - \xi_1 x_2, \\ & y_1 + y_2 \le x_1 - \xi_2, \\ & y_1 \ge 0, y_2 \ge 0. \end{aligned}$$

Do the following:

1. For any $\xi = (\xi_1, \xi_2)$, compute $K_2(\xi)$ (see the footnote[1]). **[10 points]**
2. Let $\xi = (\xi_1, \xi_2)$ take the following values with equal probability

$$\{(1,1), (2,2), (4,2)\}.$$

   Then, compute K_2 (see the footnote[2]). **[5 points]**
3. Let $\xi = (\xi_1, \xi_2)$ take values $(u, 2)$ where u follows an exponential distribution with mean value equal to one. Then, compute K_2 (see the footnote[3]). **[5 points]**

[1] Recall that this set is the set of first-stage decisions that ensure feasibility in the second stage for the particular uncertainty ξ.

[2] Recall that this is the set of first-stage decisions that ensure feasibility in the second stage for all realizations of the uncertainty ξ.

[3] The probability density function of an exponential distribution with mean equal to one is:

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Problem 4: Computing EVPI and VSS (20 points)

Consider the following two-stage problem

$$\begin{aligned} \min \quad & 2x + 7\mathbb{E}_\xi[\max\{x - \xi, -2x + \xi\}] \\ \text{subject to} \quad & 0 \leq x \leq 4. \end{aligned}$$

Note that the function inside the expectation is the max of two functions: one is $f_1(x) = x - \xi$ and the other is $f_2(x) = -2x + \xi$ (see the footnote [1]).
Assume that the random variable ξ takes two values $\xi_1 = 1$ and $\xi_2 = -1$. Let the probability distribution be given as

$$p_1 = \mathbb{P}[\xi = \xi_1] = 0.4, \text{ and } p_2 = \mathbb{P}[\xi = \xi_2] = 0.6.$$

Do the following

1. Determine the value of the wait-and-see solution (WS). **[6 points]**
2. Determine the value of the recourse problem (RP). **[6 points]**
3. Determine the expected result of using the expected value solution (EEV). **[6 points]**
4. Using the above values, determine the expected value of perfect information (EVPI) and the value of stochastic solution (VSS). **[2 points]**

[1] To elaborate more on the max of two functions, at a particular x the value that the function $f(x) = \max\{x - \xi, -2x + \xi\}$ takes is the maximum of the two values: $f_1(x)$ and $f_2(x)$. As an example, a function $g(x) = |x|$ is the max of two functions $g_1(x) = x$ and $g_2(x) = -x$.