Final Exam for Calculus 2 for IEM

Block 2A, 2021–2022

university of groningen

INSTRUCTIONS TO CANDIDATES

1. The time allowed for this exam is 2 hours.
2. Attempt all 5 questions in this test. The total number of points available is 100. You will receive 10 points for free and your score will be your point total divided by 10.
3. The number of points you can get for each question is shown next to it.
4. In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.
5. Simple pocket calculators are allowed at this exam. Other electronic devices such as graphical/programmable calculators, tablets, laptops and mobile phones are not.
6. Books, notes and formula sheets are not allowed.
7. Please write your solutions on the accompanying pages and make sure that all pages have your name and student ID on them.

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM.

$\boxed{1}$ **[15 points]**

a) Consider the function

$$\mathbf{r}(t) = \frac{1-\cos(t)}{t^2}\,\mathbf{i} + \frac{1-e^t}{t}\,\mathbf{j}.$$

Determine the limit $\lim_{t\to 0}\mathbf{r}(t)$, or explain why it does not exist.

b) Determine the following limit or explain why it does not exist:

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^4\sin(y^2)}{x^2+y^2}.$$

$\boxed{2}$ **[20 points]** Consider the surface given by $xyz + \ln yz + x^4y^2 = 2$.

a) Find a vector that is perpendicular to the surface at the point $(1, 1, 1)$.

b) The point $(1, 1, \sqrt{2})$ is expressed in terms of Cartesian coordinates. Express the point in terms of cylindrical coordinates and in terms of spherical coordinates.

c) If z can be expressed as a function of x and y, find $\frac{\partial z}{\partial x}$.

$\boxed{3}$ **[20 points]** Let $\mathbf{F} = (3x^2y + e^{x+2y}, x^3 + 2e^{x+2y})$, $x,\ y \in \mathbb{R}$.

a) Find a function $f : \mathbf{R}^2 \to \mathbb{R}$ with continuous partial derivatives such that $\nabla f = \mathbf{F}$.

b) Find the curl of $\mathbf{F}$.

c) Find the divergence of $\mathbf{F}$.

d) Suppose $f(0, 0) = 1$, where f is the function found in part a). (If you did not find f, take f to be $f(x, y) = e^x + e^{2y} - 1$). Find an equation of the plane tangent to the surface given by $z = f(x, y)$ at the point $(0, 0, 1)$.

$\boxed{4}$ **[15 points]** The position of a particle at time $t \in [0, 3]$ is given by

$$\mathbf{r}(t) = (t+4)\,\mathbf{i} + t^2\,\mathbf{j} + \frac{2t^3+1}{3}\,\mathbf{k}.$$

a) Calculate the length of the path traversed by the particle from $t = 0$ to $t = 3$.

b) Calculate the curvature of the particle's path at $t = 2$.

$\boxed{5}$ **[20 points]** Let $F(x, y) = xy + x^2 + y^2$.

a) Find all stationary points of F.

b) Find the directional derivative $D_{\mathbf{u}}F(1, 1)$ for $\mathbf{u} = \frac{1}{2}\sqrt{2}\langle 1, 1\rangle$.

c) Check for each stationary point in part a) whether it corresponds to a local maximum, a local minimum, or a saddle point.

d) Use Lagrange multipliers to find the maximum and minimum of $F(x, y)$ for the case that $x^2 + y^2 = 1$.

e) Maximise $F(x, y)$ as in d), but this time use a parametrisation $(x(t), y(t))$ of the curve given by $x^2 + y^2 = 1$ and use $\frac{d}{dt}\left(F(x(t), y(t))\right)$.