Calculus 2 (IEM)

Midterm Exam II
The 18th of March, 2022

- This exam consists of four problems, worth a total of **45** points.
- You also get five bonus points, so your total number of points will be between 5 and 50. Your final score will be your total number of points divided by 5.
- You must give complete arguments and computations and avoid leaps in logic to get full points.
- Write your **full name** and **student number** in the upper right corner of every sheet you want graded.
- Clearly mark which problem you are solving on each page.

$\boxed{1}$ Let $F(x,y) = x^5y^5 - x^3y^4 + 2022$.

a. Find $\frac{\partial^3 F}{\partial x^3}$. [**4 points**]

b. Find $\frac{\partial^2 F}{\partial y^2}$. [**4 points**]

c. Give an equation for the plane tangent to the surface given by F at the point $(1, 1, F(1,1))$. [**7 points**]

Solution:

a. $\frac{\partial^3 F}{\partial x^3} = 5 \cdot 4 \cdot 3x^2y^5 - 3 \cdot 2 \cdot 1 \cdot y^4 + 0 = 60x^2y^5 - 6y^4$.

b. $\frac{\partial^2 F}{\partial y^2} = x^5 \cdot 5 \cdot 4y^3 - x^3 \cdot 4 \cdot 3y^2 + 0 = 20x^4y^3 - 12x^3y^2$.

c. A formula for a vector that is perpendicular to the plane tangent to the surface given by F at the point $(1, 1, F(1,1))$ is

$$\begin{aligned}
&-\frac{\partial F}{\partial x}(1,1)\mathbf{i} - \frac{\partial F}{\partial y}(1,1)\mathbf{j} + \mathbf{k} = (-5 \cdot 1^4 \cdot 1^5 + 3 \cdot 1^2 \cdot 1^4)\mathbf{i} \\
&+ (-5 \cdot 1^5 \cdot 1^4 + 4 \cdot 1^3 \cdot 1^3)\mathbf{j} + \mathbf{k} \\
&= \langle -2, -2, 1 \rangle.
\end{aligned}$$

We thus find that an equation for the plane tangent to the surface given by F at the point $(1, 1, F(1,1))$ is

$$\begin{aligned}
z &= F(1,1) + \frac{\partial F}{\partial x}(1,1)(x-1) + \frac{\partial F}{\partial y}(1,1)(y-1) \\
&= 2022 + 2(x-1) + (y-1).
\end{aligned}$$

$\boxed{\mathbf{2}}$ Let

$$F(x, y, z) = (y + z) \ln x + xy^2z^3 - 4.$$

Consider the surface described by the equation $F(x, y, z) = 0$. Find a nonzero vector that is perpendicular to that surface at the point $(1, 2, 1)$ (that lies on the surface). [**4 points**]

Solution: Consider any curve on the surface through the point $(1, 2, 1)$ with parametrisation $(x(t), y(t), z(t))$. Then $F(x(t), y(t), z(t)) = 0$ and by the chain rule

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial} z \frac{dz}{dt} \\ &= \left(\frac{y+z}{x} + y^2z^3\right)\frac{dx}{dt} + \left(\ln x + 2xyz^3\right)\frac{dy}{dt} + \left(\ln x + 3xy^2z^2\right)\frac{dz}{dt}, \end{aligned}$$

which means that

$$\nabla F(1, 2, 1) = \langle 7, 4, 12\rangle$$

is perpendicular to

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

for whatever t the curve goes through $(1, 2, 1)$. As this holds true for any such curve on the surface, this means that $\nabla F(1, 2, 1) = \langle 7, 4, 12\rangle$ is perpendicular to the surface at the point $(1, 2, 1)$.

$\boxed{\mathbf{3}}$ Let $g(x, y) = xy + \frac{2}{x} + \frac{4}{y} + 10$.

a. Find all stationary points of g. [**4 points**]

b. Find the directional derivative $D_{\mathbf{u}}g(1, 1)$ for any unit vector $\mathbf{u} = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$. [**4 points**]

c. Check for each stationary point in part a. whether it corresponds to a local maximum, a local minimum, or a saddle point. [**4 points**]

a. **Solution:** The stationary points of g are those points (x, y) for which $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$.
Note that

$$\frac{\partial g}{\partial x} = y - \frac{2}{x^2} + 0 \text{ and } \frac{\partial g}{\partial y} = x + 0 - \frac{4}{y^2},$$

so $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ if and only if $y = \frac{2}{x^2}$ and $x = \frac{4}{y^2}$, if and only if $x = 1$ and $y = 2$, so the only stationary point of g is $(1, 2)$.

b. **Solution:** The directional derivative $D_{\mathbf{u}}g(1,1)$ for any unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ by part a. is

$$\begin{aligned}\nabla g(x,y)\big|_{(x,y)=(1,1)} \cdot \mathbf{u} &= \left\langle \frac{\partial g}{\partial x}(1,1), \frac{\partial g}{\partial y}(1,1) \right\rangle \cdot \langle \cos\theta, \sin\theta \rangle^T \\ &= \langle -1, -3 \rangle \cdot \langle \cos\theta, \sin\theta \rangle^T \\ &= -\cos\theta - 3\sin\theta.\end{aligned}$$

c. **Solution 1:** The only stationary point in part a. was $(1,2)$, for which we have that

$$\begin{aligned}\left.\frac{\partial^2 g}{\partial x^2}\right|_{(x,y)=(1,2)} &= \left.\frac{4}{x^3}\right|_{(x,y)=(1,2)} = 4, \\ \left.\frac{\partial^2 g}{\partial y^2}\right|_{(x,y)=(1,2)} &= \left.\frac{8}{y^3}\right|_{(x,y)=(1,2)} = 1\end{aligned}$$

and

$$\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(x,y)=(1,2)} = 1\big|_{(x,y)=(1,2)} = 1.$$

As the determinant of the Hessian is $4 \cdot 1 - 1 \cdot 1$, which is positive and as $g_{xx}(2,1) > 0$, we have that $g(1,2)$ is a local minimum value of g.
Solution 2: The only stationary point in part a. was $(1,2)$, for which we have that

$$\begin{aligned}\left.\frac{\partial^2 g}{\partial x^2}\right|_{(x,y)=(1,2)} &= \left.\frac{4}{x^3}\right|_{(x,y)=(1,2)} = 4, \\ \left.\frac{\partial^2 g}{\partial y^2}\right|_{(x,y)=(1,2)} &= \left.\frac{8}{y^3}\right|_{(x,y)=(1,2)} = 1\end{aligned}$$

and

$$\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(x,y)=(1,2)} = 1\big|_{(x,y)=(1,2)} = 1,$$

so as

$$\begin{aligned}&u^2 \left.\frac{\partial^2 g}{\partial x^2}\right|_{(x,y)=(1,2)} + 2uv \left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(x,y)=(1,2)} + v^2 \left.\frac{\partial^2 g}{\partial y^2}\right|_{(x,y)=(1,2)} \\ &= 4u^2 + 2 \cdot 1 \cdot uv + 1 \cdot v^2 \geq u^2 + 2uv + v^2 = (u+v)^2 > 0\end{aligned}$$

for all u, $v \in \mathbb{R}$, u, v not both zero, we have that $g(1,2)$ is a local minimum value of g.

$\boxed{4}$ Let $x \in \mathbb{R}$ and let $t > 0$. Let $u(x,t)$ solve the partial differential equation

$$c^2 u_{xx} - u_{tt} = 0,$$

where $c \in \mathbb{R}$ is a constant. This equation is called the *wave equation* and models many wave-like phenomena in for example physics and chemistry.

a. Let $v(\xi, \eta) = u(x,t)$, where $\xi = x + ct$ and $\eta = x - ct$. Prove that in that case $v_{\xi\eta} = 0$. [**6 points**]

b. Prove that $v(\xi, \eta) = f(\xi) + g(\eta)$ for some twice differentiable single variable functions f, g. [**6 points**]

c. Prove that $u(x,t) = f(x+ct) + g(x-ct)$, where f and g are the functions from part b. [**2 points**]

Solution:

a. **Proof:** By the chain rule we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \xi}\cdot 1 + \frac{\partial v}{\partial \eta}\cdot 1\right)$$
$$= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \xi}\cdot c - \frac{\partial v}{\partial \eta}\cdot c\right)$$
$$= c^2(v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}),$$

so the wave equation can be rewritten as

$$0 = c^2 u_{xx} - u_{tt} = c^2\left(v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}\right) - c^2(v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta})$$
$$= 4c^2 v_{\xi\eta},$$

which proves, because $c \neq 0$, that indeed $v_{\xi\eta} = 0$. *q.e.d.*

b. **Proof:** Because of part a. we have that $v_{\xi\eta} = 0$ and that the partial derivative with respect to η of v_ξ is zero, so that means that $v_\xi(\xi,\eta) = h(\xi)$ for any scalar, differentiable function $h : \mathbb{R} \to \mathbb{R}$. Because v_ξ then only depends on ξ, that means that for any scalar, differentiable function $g : \mathbb{R} \to \mathbb{R}$ we have that $v(\xi,\eta) = H(\xi) + g(\eta)$, with $H' = h$. So choosing $f = H$, we then find that indeed there are scalar, differentiable functions f, g such that $v(\xi,\eta) = f(\xi) + g(\eta)$. *q.e.d.*

c. **Proof:** Because of part b. we have that $v(\xi,\eta) = f(\xi) + g(\eta)$. As $u(x,t) = v(\xi,\eta)$, $\xi = x + ct$ and $\eta = x - ct$, this means that $u(x,t) = f(x+ct) + g(x-ct)$. *q.e.d.*