Calculus 2 (IEM)

Midterm Exam I
4 March 2022

- This exam consists of four problems, worth a total of **9** points.
- You also get one bonus point, so your final score will be **1–10**.
- You must give complete arguments and computations and avoid leaps in logic to get full points.
- Write your **full name** and **student number** in the upper right corner of every sheet you want graded.
- Clearly mark which problem you are solving on each page.

Remark: Throughout this text I will use double bars to indicate the length of a vector, i.e. for any vector $\mathbf{v}$ I will write the length of $\mathbf{v}$ as $\|\mathbf{v}\|$.

$\boxed{1}$ **(2 pt)** Consider the function

$$\mathbf{r}(t) = \frac{t}{t+\sin(t)}\,\mathbf{i} + \frac{t}{e^t - 1}\,\mathbf{j}.$$

Determine the limit $\lim_{t\to 0} \mathbf{r}(t)$ or explain why it does not exist.

Solution: By L'Hôpital's rule we have that

$$\begin{aligned}\lim_{t\to 0}\mathbf{r}(t) &= \lim_{t\to 0}\frac{t}{t+\sin(t)}\,\mathbf{i} + \frac{t}{e^t-1}\,\mathbf{j} = \lim_{t\to 0}\frac{(t)'}{(t+\sin(t))'}\,\mathbf{i} + \frac{(t)'}{(e^t-1)'}\,\mathbf{j}\\ &= \lim_{t\to 0}\frac{1}{1+\cos(t)}\,\mathbf{i} + \frac{1}{e^t}\,\mathbf{j} = \frac{1}{2}\,\mathbf{i} + \mathbf{j}.\end{aligned}$$

$\boxed{2}$ **(2 pt)** Consider the curves C_1 and C_2, respectively given by parametrizations

$$\begin{aligned}\mathbf{r}_1(t) &= (\arctan(t),\, t^2),\\ \mathbf{r}_2(u) &= (u^2 - u,\, u^2 + u).\end{aligned}$$

a) Show that $(0,0)$ is a point of intersection of C_1 and C_2.

b) Determine the angle at which C_1 and C_2 intersect at $(0,0)$.

Hint: The angle between two curves at an intersection is the angle between their tangents at that point, assuming they exist.

a) **Proof:** Note that $\mathbf{r}_1(0) = (\arctan(0),\, 0^2) = (0,0)$ and that $\mathbf{r}_2(0) = (0^2 - 0,\, 0^2 + 0) = (0,0)$. So $(0,0)$ lies on both C_1 and C_2 and we have thus proven that $(0,0)$ is a point of intersection of C_1 and C_2. *q.e.d.*

b) **Solution:** From part a) we know that $\mathbf{r}_1(0) = \mathbf{r}_2(0) = (0,0)$. Note that

$$\mathbf{r}_1'(t) = \left(\frac{1}{1+t^2},\, 2t\right) \text{ and } \mathbf{r}_2'(u) = (2u - 1,\, 2u + 1).$$

So $\mathbf{r}_1'(0) = \left(\frac{1}{1+0^2},\, 0\right) = (1,0)$ and $\mathbf{r}_2'(0) = (0 - 1,\, 0 + 1) = (-1,1)$. As $\mathbf{r}_1'(0)$ has length 1 and $\mathbf{r}_2'(0)$ has length $\sqrt{2}$, we have that if θ is the angle at which C_1 and C_2 intersect at $(0,0)$, then $\cos\theta = \frac{1}{\sqrt{2}}\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = -\frac{1}{\sqrt{2}}$, so $\theta = \frac{3}{4}\pi$. So the angle at which C_1 and C_2 intersect at $(0,0)$ is $\frac{3}{4}\pi$.

$\boxed{3}$ **(3 pt)** The position of a particle at time $t \in [1,4]$ is given by

$$\mathbf{r}(t) = (t^2+4t)\,\mathbf{i} + (t^2-4t)\,\mathbf{j} + \frac{t^3+6t}{3}\,\mathbf{k}.$$

Determine the following:

a) The length of the path traversed by the particle.

b) The curvature of the particle's path at $t=2$.

c) The tangential and normal components of acceleration of the particle at $t=2$.

a) **Solution:** The length of the path traversed by the particle is

$$\begin{aligned}\int_1^4 \|\mathbf{r}'(t)\|dt &= \int_1^4 \sqrt{\left((t^2+4t)'\right)^2 + \left((t^2-4t)'\right)^2 + \left(\left(\frac{t^3+6t}{3}\right)'\right)^2}\,dt \\ &= \int_1^4 \sqrt{(2t+4)^2 + (2t-4)^2 + (t^2+2)^2}\,dt \\ &= \int_1^4 \sqrt{t^4+12t^2+36}\,dt = \int_1^4 \sqrt{(t^2+6)^2}\,dt \\ &= \int_1^4 (t^2+6)dt = \left[\frac{1}{3}t^3+6t\right]_1^4 = 21+18 = 39.\end{aligned}$$

b) **Solution:** The curvature κ of the particle's path at t is by the calculation in part a) equal to

$$\begin{aligned}\kappa &= \frac{\|\mathbf{r}''(t)\times\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{1}{\|\mathbf{r}'(t)\|^3}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t+4 & 2t-4 & t^2+2 \\ 2 & 2 & 2t\end{vmatrix}\right\| \\ &= \frac{1}{(t^2+6)^3}\left((2t^2-8t-4)^2 + (2t^2+8t-4)^2 + (16)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{(t^2+6)^3}\left(8(t^2+6)^2\right)^{\frac{1}{2}} = \frac{2\sqrt{2}}{(t^2+6)^2}.\end{aligned}$$

So the curvature of the particle's path at $t=2$ is

$$\frac{2\sqrt{2}}{(2^2+6)^2} = \frac{1}{50}\sqrt{2}.$$

c) The tangential component of the acceleration is by the calculation in part a) and b) equal to

$$\frac{\mathbf{r}''(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2(2t+4)+2(2t-4)+2t(t^2+2)}{t^2+6} = 2t.$$

and the normal component of the acceleration is by the calculation in part a) and b) equal to

$$\frac{\|\mathbf{r}''(t)\times\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|} = 2\sqrt{2}.$$

so the tangential and normal components of acceleration of the particle at $t=2$ are

$$\frac{\mathbf{r}''(2)\cdot\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = 4 \text{ and } \frac{\|\mathbf{r}''(2)\times\mathbf{r}'(2)\|}{\|\mathbf{r}'(t)\|} = 2\sqrt{2}$$

respectively.

$\boxed{4}$ **(2 pt)** Determine the following limit or explain why it does not exist:

a) $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3\sin(y)}{x^2+y^4}$, b) $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+xy+y^3}{x^3-xy+y^3}$.

Solution:

a) We have that

$$\left|\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3\sin(y)}{x^2+y^4}\right| = \lim_{\substack{x\to 0\\ y\to 0}} \left|\frac{x^3\sin(y)}{x^2+y^4}\right| \le \lim_{\substack{x\to 0\\ y\to 0}} \frac{|x^3|\cdot|\sin y|}{x^2+0}$$
$$\le \lim_{\substack{x\to 0\\ y\to 0}} \frac{|x^3|\cdot 1}{x^2} = \lim_{\substack{x\to 0\\ y\to 0}} |x| = 0.$$

So $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3\sin(y)}{x^2+y^4} = 0.$

b) First let $y=x$. Then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+xy+y^3}{x^3-xy+y^3} = \lim_{x\to 0} \frac{x^3+x^2+x^3}{x^3-x^2+x^3} = -1.$$

Secondly, let $y=x^2$. Then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+xy+y^3}{x^3-xy+y^3} = \lim_{x\to 0} \frac{x^3+x^4+x^6}{x^3-x^4+x^6} = 1.$$

So the limit does not exist.