Calculus 2 Summary

Lars Hof

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Contents

1 Vectors and the geometry of space

To compute the distance between two points, one can use the Distance formula in three dimensions, where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{Distance formula in three dimensions}$$

1.1 Vectors

A vector is denoted by $\vec{a} = (a_1, a_2, ..., a_n)$, where $(a_1, a_2, ..., a_n)$ are the components of $\vec{a}$ and n is the number of dimensions.

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2} \qquad \text{Length of a vector}$$

The **standard basis vector** has length 1, and point in positive direction on all axes.

$$\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \frac{\vec{a}}{|\vec{a}|} \qquad \text{Unit vector}$$

1.2 The Dot Product

Take $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then **the dot product** is determined by the Dot product equation.

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{Dot product}$$

- If θ is the angle between two vectors and , then: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$
- Two vectors and are **orthogonal** if and only if $\vec{a} \cdot \vec{b} = 0$
- The **direction angles** of a vector can be determined by using the Direction angles equation. Where α represents the angle to the x-axis, β represents the angle to the y-axis, and γ represents the angle to the z-axis

$$\frac{1}{|\vec{a}|}\vec{a} = (\cos\alpha, \cos\beta, \cos\gamma) \qquad \text{Direction angles}$$

1.3 The cross product

The cross product, or vector product is specific to R and is defined as

$$\vec{a} \times \vec{b} = \mathbf{det} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \qquad \text{Cross product}$$

- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$
- if θ is the angle between $\vec{a}$ and $\vec{b}$, then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.
  In other words, the length of $\vec{a} \times \vec{b}$ is the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$
- Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$

1.3.1 Properties of the cross product

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $c\,\vec{a} \times \vec{b} = c(\vec{a} \times \vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\vec{a} + \vec{b}) \times \vec{c}) = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \times \vec{b} \cdot \vec{c}$
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

1.3.2 Triple products

The triple product describes the volume of a **parallelepiped**. determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

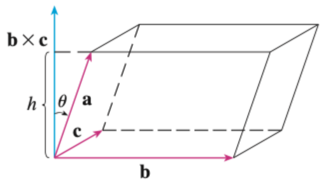


Figure 1: Visualization of a paralellepiped depending on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

1.4 Equations of lines and planes

To desribe a line in R^3 one can use the Vector equation, where r_0 describes a fixed point, and $\vec{v}$ is multiplied with any real scalar.

$$\mathbf{r} = \mathbf{r}_0 + \alpha\vec{v} \qquad \text{Vector equation}$$

1.5 Cylinders and Quadric Surfaces

We distinguish six kinds of quadric surfaces as defined in figure 3

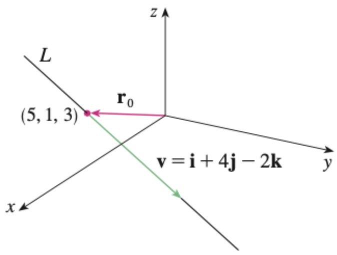
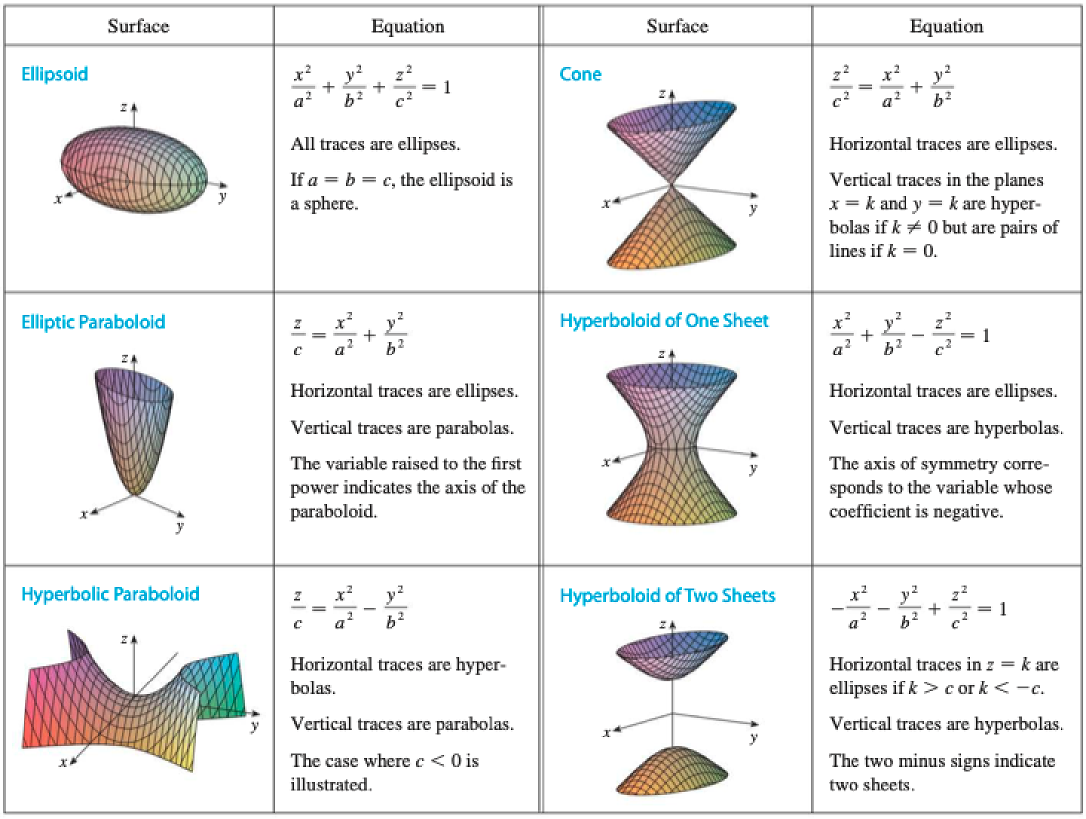


Figure 2: Visualization of vector equation in R^3



Surface	Equation	Surface	Equation
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Figure 3: Graphs of quadric surfaces

2 Vector Functions

A **vector-valued function** is simply a function whose domain is a set of real numbers and whose range is a set of vectors. $f(t)$, $g(t)$, and $h(t)$ are called the **component functions** of the vector function $\mathbf{r}(t)$.

$$\mathbf{r}(t) = (f(t), g(t), h(t)) \tag{1}$$

2.1 Limits and continuity

The **limit** of a vector function is defined by taking the limit of its component functions.

$$\lim_{t\to a}\mathbf{r}(t) = \left(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right) \tag{2}$$

2.2 Derivatives

The derivative of a vector function is similar to the normal definition of a derivative.

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{3}$$

Conveniently, this means that the derivative of a vector function is obtained by differentiating each component.

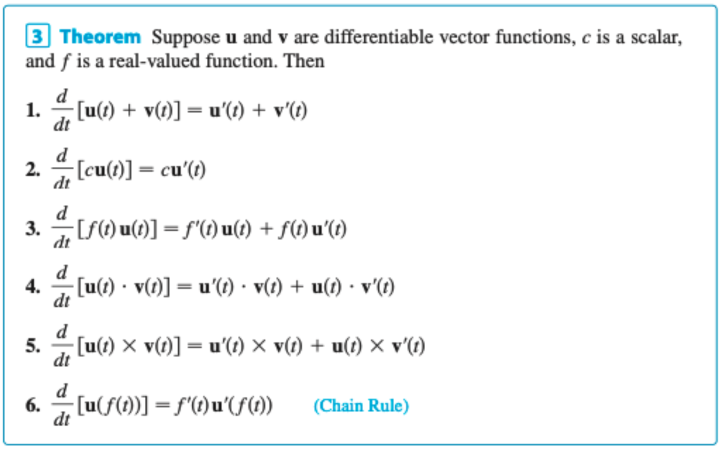
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \big(f'(t), g'(t), h'(t)\big) \tag{4}$$

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the vector function. The unit tangent vector is determined as follows:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{5}$$

2.2.1 Differentiation rules

The following theorem shows that the differentiation rules for real-valued functions have their counterparts for vector functions.


3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Figure 4: Differentiation rules for vector functions

2.2.2 Integrals

The **definite integral** of a continues vector function can be defined in almost the same way as for real-valued functions, except that the integral is a vector.

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right) \tag{6}$$

2.3 Arc length and Curvature

The length of a space curve is defined in equation 7

$$L = \int_a^b |\mathbf{r}'(t)|\;dt \tag{7}$$

A curve where a certain segment of a vector equation is only traversed once can be expressed by the **arc length function** defined in equation 8

$$\mathbf{s}(t) = \int_a^t |\mathbf{r}'(u)| \; dt \tag{8}$$

Equation 8 can be useful to **parametrize a curve with respect to the arc length**. If we want to reparametrize a vector equation from a specific point u we find $s(t)$ using equation 8 and then insert $s(t)$ into the original vector equation. (See example 2 in the book, page 863)

2.3.1 Curvature

The curvature of a vector equation at a given point is the measure of how quickly the curve changes direction at that point. The curvature can be computed through either equation 9, or equation 10.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{9}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{10}$$

2.3.2 The normal and binormal vectors

The **unit normal vector** describes at which direction the curve is headed at each point, and can be computed as defined in equation 11. The unit normal vector at t is used to describe a normal plane at t.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \tag{11}$$

The **binormal vector** is a vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$. And can be computed with equation 12. The binomal vector at t is used to describe an oscillating plane at t.

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \tag{12}$$

2.4 Motion in Space: Velocity and Acceleration

Vector $\mathbf{v}$ gives the average velocity over a time interval of length h and it's limit is the **velocity vector** at time t

$$\mathbf{v} = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t) \tag{13}$$

The speed at time t is defined by the magnitude of the velocity vector.

$$\frac{ds}{dt} = |\mathbf{v}(t)| = |\mathbf{r}'(t)| \tag{14}$$

The **accelatrion** of a particle is defined as the derivative of the velocity.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \tag{15}$$

2.4.1 Tangential and Normal Components of Acceleration

It is useful to resolve the acceleration in two components, one in the direction of the tangent vector, and one in direction of the normal vector, as described in equation

$$\mathbf{a} = \upsilon'\mathbf{T} + \kappa\upsilon^2\mathbf{N} \tag{16}$$

The tangential and normal components can be easily calculated with the following two formula's:

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \tag{17}$$

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \tag{18}$$

2.4.2 Useful formulas

$$\mathbf{F}(t) = m\mathbf{a}(t) \tag{19}$$

Where, F represents force, m represents mass, a represents acceleration.

$$\mathbf{v}(t) = \int_{t_0}^{t} \mathbf{a}(u)du + \mathbf{v}(t_0) \tag{20}$$

Where v represents the velocity at a time t, u represents the initial point in space, a represents acceleration. $\mathbf{v}(t_0)$ represents the initial velocity.

$$d = \frac{{v_0}^2 \sin(2\theta)}{g} \tag{21}$$

The horizontal distance traveled is determined by the product of the horizontal speed and the duration of travel.

3 Partial Derivatives

3.1 Functions of several variables

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and it's **range** is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D\}$

3.1.1 Graphs

If f is a function of two variables with domain D, then the graph of f is the set of all points $(x, y, z) \in^3$ such that $z = f(x, y)$, and $(x, y) \in D$

3.2 Limits and Continuity

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) Then we say that the limit of $f(x, y)$ as (x, y) approaches $(a, b) = L$ and we write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{22}$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that, if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

Notice that $|f(x, y)-L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

If $\lim_{(x,y)\to(a,b)} f(x, y)$ approaches L_1 along a certain path, and $\lim_{(x,y)\to(a,b)} f(x, y)$ approaches L_2 along a different path, where $L_1 \neq L_2$, the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

A number of ways of approaching whether the limits exists or not are:

- Approaching along the x- and y-axis
- Approaching along the line $y = x$
- Approaching along the line $y = ax$
- (Polar) Substitution

3.2.1 Continuity

A **function** $f(x, y)$ is **continues** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

3.2.2 Functions of three or more variables

If f is defined on a subset $D \in^n$, then $\lim_{x\to a} f(x) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that: if $x \in D$, and, $0 < |a - x| < \delta$, then $|f(x) - L| < \epsilon$

3.3 Partial Derivatives

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$.

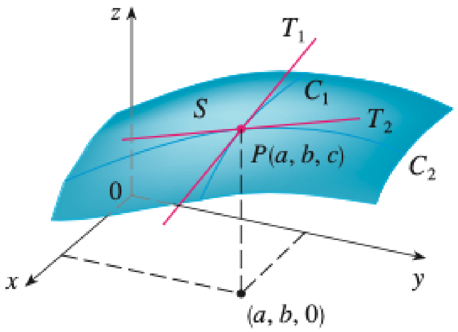
$$f_x(a, b) = g'(a) \text{ where } g(x) = f(x, b) \tag{23}$$

Since by definition $g'(a) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ we get

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{24}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} \tag{25}$$

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = a$ and $x = b$.



3.3.1 Notation for partial derivatives

If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x}$$

3.3.2 Rule for finding partial derivative of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y

3.3.3 Function of more than two variables

$$f_x(x, y, z) = \lim h \to 0 \frac{f(x + h, y, z) - f(x, y, z)}{h} \tag{26}$$

3.3.4 Higher derivatives

Higher derivatives can be taken with respect to any variable and are denoted as follows:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

3.3.5 Clairaut's Theorem

If $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are both defined and $f(a, b)$ is continues, equation 27 holds.

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{27}$$

3.3.6 Laplace's equation

A function $f(a, b)$ is **harmonic** if equation 28 holds.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0 \tag{28}$$

3.4 Tangent Planes and Linear Approximations

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{29}$$

3.4.1 Linear Approximations

The linear equation that follows from equation 29 is called the **linearization** of the tangent plane to the graph of a function f of two variables at the point (a,b,f(a,b)) is

$$L(x, y) = f_x(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The **Linear Approximation** is defined as follows:

$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

3.4.2 Differentials

For a differentiable function of two variables, $z(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined in equation 30

$$dz = f_x(x, y)dx + f_y(x, y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{30}$$

3.5 The chain rule

Suppose that $z = f(x, y)$, is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{31}$$

The derivative $\frac{dz}{dt}$ can be interperated as the rate of change of z with respect to t.

We now consider the situation where $z = f(x, y)$ but x and y are both a function of two variables $x = g(t, s)$ and $y = h(t, s)$. Then z is indirectly a function of s and t and we wish to find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$. In computing $\frac{\partial z}{\partial t}$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore we can apply equation 31 to obtain:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

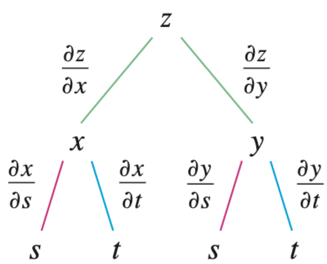


Figure 5: Overview of different partials when $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$

In this situation, s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

3.5.1 Implicit differentiation

The chain rule can simplify implicit differentiation as learned in calculus 1. For a function that is defined implicitly it follows that:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{32}$$

A function of 3 variables that is defined implicitly can also be differentiated as follows

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

3.6 Directional Derivatives and the gradient vector

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (a, b)$ and:

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b \tag{33}$$

3.6.1 The Gradient Vector

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined as:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{34}$$

Moreover, the rate of change can be defined as:

$$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u} \tag{35}$$

Similairly, for a function of more than 2 variables:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{36}$$

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u} \tag{37}$$

3.6.2 Maximizing the directional derivative

The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$, which occurs when $\mathbf{u}$ has the same direction as the gradient vector.

3.6.3 Tangent planes to level surfaces

If F is a surface of a function of three variables $F(x,y,z) = k$, P is a point on this surface $P(x_0, y_0, z_0)$ corresponding to parameter value t_0, C is any vector function $\mathbf{r}'(t) = (x(t), y(t), z(t))$, the equation of the tangent plane at point P is defined as:

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

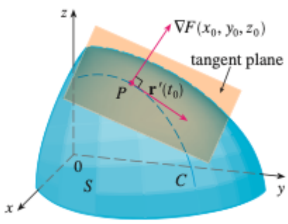


Figure 6: Tangent plane to a function of three variables

Using the standard equation of the tangent plane as in equation 29 we get:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

The **normal line** to S at P can be described is defined as:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} + \frac{y-y_0}{F_y(x_0,y_0,z_0)} + \frac{z-z_0}{F_z(x_0,y_0,z_0)} = 0 \tag{38}$$

3.7 Maximum and Minimum values

To determine the local minimum and maximum of a function of two or more variables we first have to determine the **critical point**.

- The critical point exist at (a, b) when $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Subsequently, to determine whether the critical point is a maximum, minimum, or saddle point (neither), a second-derivative test is performed. Let:

- Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$

1. $D > 0$ and $f_{xx}(a, b) > 0$ (a, b) is a **local minimum**
2. $D > 0$ and $f_{xx}(a, b) < 0$ (a, b) is a **local maximum**
3. $D < 0$, (a, b) is a **saddle point**

3.8 Langrange multipliers

Another way to find maximum and minimum values is by Langrange multipliers. Or in other words, to find the maximum and minimum values of $f(x, y, z)$ constrainted by some $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$.

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

   and

$$g(x, y, z) = k$$

2. Evaluate f at all points (x, y, z) that result from step a. Largest values are maximum and smallest are minimum.

4 Vector Calculus

4.1 Vector Fields

Let D be a set in $\mathbb{R}^n$. A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns each point $(x, y, ..., n)$ a n-dimensional vector $\mathbf{F}(x, y, ..., n)$

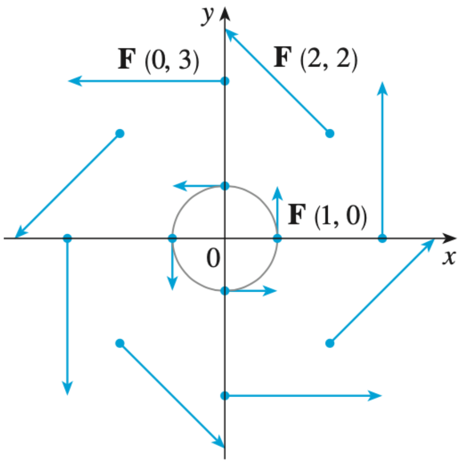


Figure 7: Example of a vector field

4.1.1 Gradient Fields

From equation 34 we know that $\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ which is really a vector field on $\mathbb{R}^2$. In a gradient vector field, the vectors are perpendicular to the level curves.

4.2 Line integrals

In a line integral, instead of over an interval $[a, b]$ we integrate over a curve C.

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 \left(\frac{dy}{dt}\right)^2} dt \tag{39}$$

Where $x(t)$ and $y(t)$ are parametrizations with respect to some t and ds is the length of curve C.

Sometimes it can be hard to find a parametrization for the curve. In this case we can also use vector representation.

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \leqslant t \leqslant 1$$

However, instead of integrating over some interval $[a, b]$ we integrate with respect to t and thus, from 0 to 1.

4.2.1 Line integrals in space

Line integrals in space are similar to equation 39, just with three variables. They can also be denoted in vector notation:

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

4.2.2 Line integrals of Vector Fields

We can determine the **work** W done by a vector field by finding the integral of the dot product of the force field $\mathbf{F}$ and the unit tangent vector $\mathbf{T}$ at a given point

$$W = \int_C \mathbf{F}(x,y,z) \cdot \mathbf{T}(x,y,z)ds = \int_C \mathbf{F} \cdot \mathbf{T}ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

4.3 The fundamental theorem for line integrals

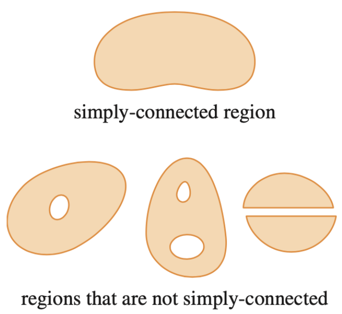
Let C be a smooth curve given by the vector function $\mathbf{r}$(t), $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{40}$$

4.3.1 Independence of path

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 that have the same initial point and terminal point. It now follows that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Suppose $\mathbf{F}$ is a vector field on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a **conservative** vector field on D. That is, if there exists a function f such that $\nabla f = \mathbf{F}$



Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose P and Q have continuous first-order partial derivatives. If:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D, then $\mathbf{F}$ is conservative.

4.4 Curl and Divergence

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is vector field on $\mathbb{R}^3$ and the partial derivatives all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\texttt{curl}\ \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Or

$$\texttt{curl}\ \mathbf{F} = \nabla \times \mathbf{F}$$

- If f is a function of three variables with second order partial derivatives, then: $\texttt{curl}(\nabla f) = 0$
- If $\texttt{curl}\mathbf{F} = 0$ then $\mathbf{F}$ is conservative.

4.4.1 Divergence

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and the partial derivatives all exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by:

$$\texttt{div}\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Or

$$\texttt{div}\ \mathbf{F} = \nabla \cdot \mathbf{F}$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is vector field on $\mathbb{R}^3$ and the second-order partial derivatives all exist, then

$$\texttt{div}\ \texttt{curl}\ \mathbf{F} = 0$$