Calculus 2 (IEM)

Final Exam

7 April 2021, 15:00–18:30

Problem 1 (1.5 points)

Given is the curve parametrized by $\mathbf{r}\colon [-1,1] \to \mathbb{R}^3$, where

$$\mathbf{r}(t) = 3t^2\,\mathbf{i} + (3t^3 - 5t)\,\mathbf{j} + 6t^2\,\mathbf{k}.$$

a) At each point of the curve, determine the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$.

b) Determine the length of the curve.

Solution

a) Differentiating gives

$$\mathbf{r}'(t) = 6t\,\mathbf{i} + (9t^2 - 5)\,\mathbf{j} + 12t\mathbf{k},$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{36t^2 + 81t^4 - 90t^2 + 25 + 144t^2} = \sqrt{81t^4 + 90t^2 + 25} \\ &= \sqrt{(9t^2+5)^2} = 9t^2 + 5. \end{aligned}$$

Therefore the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{6t}{9t^2+5}\,\mathbf{i} + \frac{9t^2-5}{9t^2+5}\,\mathbf{j} + \frac{12t}{9t^2+5}\,\mathbf{k}.$$

Differentiating $\mathbf{T}(t)$ we get (by the quotient rule)

$$\mathbf{T}'(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{6(9t^2-5)}{(9t^2+5)^2}\,\mathbf{i} + \frac{180t}{(9t^2+5)^2}\,\mathbf{j} - \frac{12(9t^2-5)}{(9t^2+5)^2}\,\mathbf{k}$$

and

$$\begin{aligned} |\mathbf{T}'(t)| &= \frac{\sqrt{36(9t^2-5)^2 + 180^2t^2 + 144(9t^2-5)^2}}{(9t^2+5)^2} \\ &= \frac{\sqrt{180(9t^2-5)^2 + 180^2t^2}}{(9t^2+5)^2} = \frac{6\sqrt{5}\sqrt{81t^4+180t^2+25}}{(9t^2+5)^2} \\ &= \frac{6\sqrt{5}\sqrt{(9t^2+5)^2}}{(9t^2+5)^2} = \frac{6\sqrt{5}}{9t^2+5}. \end{aligned}$$

Therefore

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\frac{1}{\sqrt{5}} \cdot \frac{9t^2 - 5}{9t^2 + 5}\,\mathbf{i} + 6\sqrt{5} \cdot \frac{t}{9t^2 + 5}\,\mathbf{j} - \frac{2}{\sqrt{5}} \cdot \frac{9t^2 - 5}{9t^2 + 5}\,\mathbf{k},$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{5}}{9t^2 + 5} \cdot \frac{1}{9t^2 + 5} = \frac{6\sqrt{5}}{(9t^2 + 5)^2}.$$

b) The length of C is

$$\begin{aligned} \ell(C) &= \int_C \mathrm{d}s = \int_{-1}^{1} |\mathbf{r}'(t)|\,\mathrm{d}t = \int_{-1}^{1} (9t^2 + 5)\,\mathrm{d}t \\ &= \big(3t^3 + 5t\big)\Big]_{-1}^{1} = 2\,(3 + 5) = 16. \end{aligned}$$

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Problem 2 (1.5 points)

Given is the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^3 - x^3y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) Show that f is continuous at $(0,0)$.

b) Determine whether f is differentiable at $(0,0)$.

Solution

a) Set $(x,y) = (r\cos\theta, r\sin\theta)$ to get

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^2y^3 - x^3y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^2y^2(x-y)}{x^4+y^4} \\ &= \lim_{r\to 0^+} \frac{r^5\cos^2\theta\sin^2\theta(\cos\theta - \sin\theta)}{r^4(\cos^4\theta + \sin^4\theta)} \\ &= \lim_{r\to 0^+} r\frac{\cos^2\theta\sin^2\theta(\cos\theta - \sin\theta)}{\cos^4\theta + \sin^4\theta} = 0, \end{aligned}$$

by the squeeze theorem. In particular, for the function

$$\frac{\cos^2\theta\sin^2\theta(\cos\theta - \sin\theta)}{\cos^4\theta + \sin^4\theta}$$

the numerator is bounded between -2 and 2 and the denominator is at least $\frac{1}{2}$, so the function is bounded. (In fact, it oscillates between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$). Since

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0),$$

we conclude that f is continuous.

b) We find the partial derivatives of f at $(0,0)$ by definition:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0}{x^5} = 0,$$
$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0}{y^5} = 0.$$

Therefore the linearization of f at $(0,0)$ is the function

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0.$$

To check differentiability, we compute

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{|(x,y)-(0,0)|} &= \lim_{(x,y)\to(0,0)} \frac{x^2y^2(x-y)}{(x^4+y^4)\sqrt{x^2+y^2}} \\
&= \lim_{r\to 0^+} \frac{r^5\cos^2\theta\sin^2\theta(\cos\theta-\sin\theta)}{r^5(\cos^4\theta+\sin^4\theta)} \\
&= \lim_{r\to 0^+} \frac{\cos^2\theta\sin^2\theta(\cos\theta-\sin\theta)}{\cos^4\theta+\sin^4\theta}.
\end{aligned}$$

The limit is clearly indefinite because it depends on θ, so in particular it is not zero and we conclude that f is not differentiable.

Alternatively, we can set $(x,y) = (t,2t)$ with $t \to 0$ to see that the limit is not defined.

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Problem 3 (1.5 points)

a) Show that the equation

$$x\sin(x) + y\sin(y) + z\sin(z) = 0,$$

defines a function $x = f(y,z)$ for values of (x,y,z) that are close enough to the point $(-\pi,0,0)$.

b) Show that the point $(y_0,z_0) = (0,0)$ is an extremum of $f(y,z)$ and determine its type.

SOLUTION

a) Set $F(x,y,z) = x\sin(x) + y\sin(y) + z\sin(z)$. Clearly $F(-\pi,0,0) = 0$ so the given point satisfies the equation. Since $F(x,y,z)$ is infinitely differentiable, we have

$$\nabla F = (x\cos(x) + \sin(x),\ y\cos(y) + \sin(y),\ z\cos(z) + \sin(z)).$$

In particular, for $(x_0,y_0,z_0) = (-\pi,0,0)$ we have

$$\begin{aligned} F_x(-\pi,0,0) &= -\pi\cos(-\pi) + \sin(-\pi) = \pi, \\ F_y(-\pi,0,0) &= 0\cos(0) + \sin(0) = 0, \\ F_z(-\pi,0,0) &= 0\cos(0) + \sin(0) = 0. \end{aligned}$$

Since the partial derivatives F_x, F_y, F_z are continuous, the Implicit Function Theorem implies that $x = f(y,z)$ when $(x,y,z) \approx (-\pi,0,0)$ for some differentiable function f such that

$$\frac{\partial f}{\partial y} = -\frac{F_y}{F_x}, \qquad \frac{\partial f}{\partial z} = -\frac{F_z}{F_x}.$$

b) In particular we have

$$\frac{\partial x}{\partial y}(0,0) = 0, \qquad \frac{\partial x}{\partial z}(0,0) = 0.$$

In other words $\nabla x(0,0) = (0,0)$ so $(0,0)$ is a critical point. To determine its type we will apply the Hessian criterion. We have

$$\frac{\partial x}{\partial y} = -\frac{y\cos(y) + \sin(y)}{x\cos(x) + \sin(x)}. \tag{1}$$

Differentiating (1) with respect to y and applying the quotient rule gives

$$\begin{aligned}\frac{\partial^2 x}{\partial y^2} = &-\frac{-y\sin(y) + 2\cos(y)}{x\cos(x) + \sin(x)} \\ &+ \frac{(y\cos(y) + \sin(y))(-x\sin(x) + 2\cos(x))\frac{\partial x}{\partial y}}{(x\cos(x) + \sin(x))^2}.\end{aligned}$$

Since $\dfrac{\partial x}{\partial y}(0,0) = 0$ this gives

$$\begin{aligned}\frac{\partial^2 x}{\partial y^2}(0,0) &= \left(\frac{y\sin(y) - 2\cos(y)}{x\cos(x) + \sin(x)}\right)(-\pi, 0, 0) \\ &= \frac{0\sin(0) - 2\cos(0)}{-\pi\cos(-\pi) + \sin(-\pi)} = -\frac{2}{\pi}.\end{aligned}$$

Differentiating (1) with respect to z gives

$$\frac{\partial^2 x}{\partial z \partial y} = (y\sin(y) - 2\cos(y))\frac{-(-x\sin(x) + 2\cos(x))\frac{\partial x}{\partial z}}{(x\cos(x) + \sin(x))^2}.$$

Since $\dfrac{\partial x}{\partial z}(0,0) = 0$ this gives $\dfrac{\partial^2 x}{\partial z \partial y}(0,0) = 0$. Since by a) we have

$$\frac{\partial x}{\partial z} = -\frac{z\cos(z) + \sin(z)}{x\cos(x) + \sin(x)},$$

the exact same argument gives

$$\frac{\partial^2 x}{\partial^2 z} = -\frac{2}{\pi}, \qquad \frac{\partial^2 x}{\partial y \partial z} = 0.$$

It follows that the Hessian of $x = f(y,z)$ at $(y_0, z_0) = (0,0)$ is

$$H_f(0,0) = \begin{pmatrix} -\frac{2}{\pi} & 0 \\ 0 & -\frac{2}{\pi} \end{pmatrix}.$$

Therefore $d_1 = f_{yy}(0,0) = -\frac{2}{\pi} < 0$ and $d_2 = \det H_f(0,0) = \frac{4}{\pi^2} > 0$ and by the second derivative test (the Hessian criterion) we have that $(0,0)$ is a local maximum for $f(y,z)$.

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Problem 4 (1.5 points)

Let $E \subset \mathbb{R}^2$ be the ellipse $x^2+2y^2 = 3$ and let $f(x, y) = x^2y^2+x$. Determine the points of E for which f achieves its maximum and the points of E for which f achieves its minimum.

Solution

Set $g(x, y) = x^2+2y^2-3$. We are looking for extrema of $f(x, y) = x^2y^2+x$ subject to the constraint $g(x, y) = 0$ and we will find them using Lagrange multipliers. We find the two gradients:

$$\nabla f = (2xy^2 + 1, 2x^2y), \qquad \nabla g = (2x, 4y).$$

Note that $\nabla g \neq \mathbf{0}$ for all $(x, y) \in E$, so we only need to find solutions (x, y, λ) to the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$. The system can be written as

$$2xy^2 + 1 = 2\lambda x, \tag{1}$$
$$x^2y = 2\lambda y, \tag{2}$$
$$x^2 + 2y^2 = 3. \tag{3}$$

Note that $x = 0$ contradicts (1). Suppose that $y = 0$. Then from (3) we have $x = \pm\sqrt{3}$ and we have our first pair of critical points, namely

$$P_{1,2} = \left(\pm\sqrt{3}, 0\right).$$

Suppose instead that $y \neq 0$. Then (2) gives $2\lambda = x^2$ and substituting this into (1) gives

$$2xy^2 + 1 = x^3 \Leftrightarrow y^2 = \frac{x^3 - 1}{2x}.$$

Substituting this into (3) gives

$$x^2 + \frac{x^3 - 1}{x} - 3 = 0 \Leftrightarrow x^3 + x^3 - 1 - 3x = 2x^3 - 3x - 1 = 0.$$

Clearly $x = -1$ is a solution so we can factor and rewrite as

$$(x+1)(2x^2 - 2x - 1) = 0.$$

Equating the second factor with zero gives

$$x = \frac{1 \pm \sqrt{3}}{2}.$$

Since these three solutions satisfy $2x^3 = 3x + 1$ we can write

$$y^2 = \frac{2x^3 - 2}{4x} = \frac{3x - 1}{4x}$$

and therefore we obtain

$$\begin{aligned} x = -1 &\implies y^2 = 1, \\ x = \frac{1+\sqrt{3}}{2} &\implies y^2 = \frac{4-\sqrt{3}}{4}, \\ x = \frac{1-\sqrt{3}}{2} &\implies y^2 = \frac{4+\sqrt{3}}{4}. \end{aligned}$$

This is enough to evaluate $f(x, y)$ because the function depends only on y^2. The actual critical points are

$$\begin{aligned} P_{3,4} &= (-1, \pm 1), \\ P_{5,6} &= \left(\frac{1+\sqrt{3}}{2}, \pm \frac{\sqrt{4-\sqrt{3}}}{2} \right), \\ P_{7,8} &= \left(\frac{1-\sqrt{3}}{2}, \pm \frac{\sqrt{4+\sqrt{3}}}{2} \right) \end{aligned}$$

and we have

$$\begin{aligned} f\left(\pm\sqrt{3}, 0\right) &= \pm\sqrt{3} \approx \pm 1.73205, \\ f(-1, \pm 1) &= 1 - 1 = 0, \\ f\left(\tfrac{1+\sqrt{3}}{2}, \pm\tfrac{\sqrt{4-\sqrt{3}}}{2}\right) &= \tfrac{(1+\sqrt{3})^2}{4} \cdot \tfrac{4-\sqrt{3}}{4} + \tfrac{1+\sqrt{3}}{2} = \tfrac{3}{8}(3 + 2\sqrt{3}) \approx 2.42404, \\ f\left(\tfrac{1-\sqrt{3}}{2}, \pm\tfrac{\sqrt{4+\sqrt{3}}}{2}\right) &= \tfrac{(1-\sqrt{3})^2}{4} \cdot \tfrac{4+\sqrt{3}}{4} + \tfrac{1-\sqrt{3}}{2} = \tfrac{3}{8}(3 - 2\sqrt{3}) \approx -0.174038. \end{aligned}$$

Hence the maximum is achieved at $\left(\frac{1+\sqrt{3}}{2}, \pm\frac{\sqrt{4-\sqrt{3}}}{2}\right) \approx (1.366, \pm 0.753)$ and that the minimum is achieved at $(-\sqrt{3}, 0) \approx (1.732, 0)$.

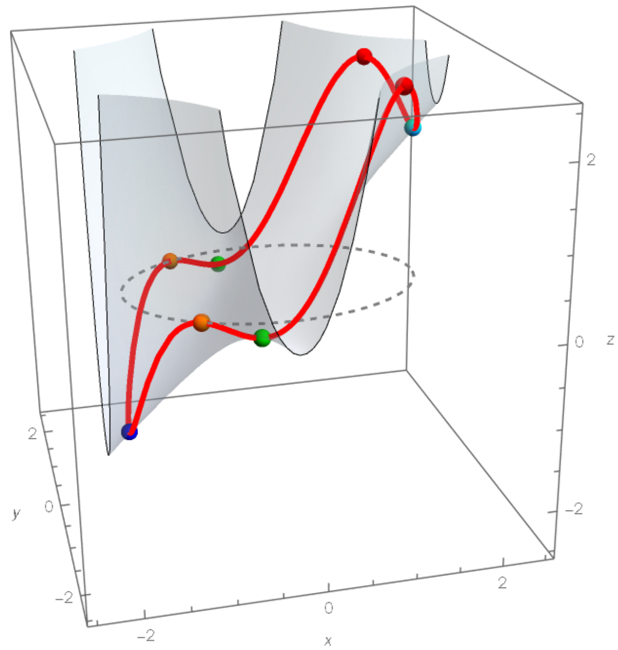

Figure 1: The graph of f, the ellipse E and its image on the graph, and the critical points

Notice in what sense the critical points are special. If $(x(t), y(t))$ is a parametrization of the ellipse E then the curve

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k}.$$

is on the graph (coloured red in the Figure) and it has horizontal tangents for those values of t for which $(x(t), y(t))$ is a critical point, so this is very similar to the first derivative test for functions of one variable. The good thing about the method of Lagrange multipliers is that we do not need to parametrize the level curve; the equation $g(x, y) = 0$ suffices.

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Problem 5 (1.5 points)

a) Describe the curve $y = x^2$ using polar coordinates (r, θ), i.e. write r as a function of θ (or the other way around). Be sure to specify the domain of the independent variable.

b) Given is a quadric surface whose equation in cylindrical coordinates (r, θ, z) is

$$r^2(1 + \cos^2\theta) = z^2 + 1$$

Describe this surface using Cartesian coordinates and determine which type it is.

SOLUTION

a) Substituting $x = r\cos\theta$ and $y = r\sin\theta$ with $r \geq 0$ and $\theta \in [0, 2\pi)$ gives

$$r\sin\theta = r^2\cos^2\theta.$$

Therefore for $(x, y) \neq (0, 0)$, i.e. $r \neq 0$ we have

$$r = \frac{\sin\theta}{\cos^2\theta}. \tag{1}$$

The curve is a parabola in the upper half-plane and each of its points can be thought of as a vector $\boldsymbol{v}(t) = t\mathbf{i} + t^2\mathbf{j}$. The cosine of its angle with the x-axis is

$$\frac{\boldsymbol{v}(t) \bullet \mathbf{i}}{|\boldsymbol{v}(t)|} = \frac{(t\mathbf{i} + t^2\mathbf{j}) \bullet \mathbf{i}}{|t\mathbf{i} + t^2\mathbf{j}|} = \frac{t}{\sqrt{t^2 + t^4}} = \frac{t}{|t|\sqrt{1 + t^2}},$$

which approaches 0 as $t \to \infty$ (so the angle approaches $\frac{\pi}{2}$). Another way of seeing this is by looking at the denominator in (1) – it approaches zero as $\theta \to \frac{\pi}{2}$. As $t \to 0^{\pm}$, the cosine of the angle with the

x-axis approaches ± 1 (so the angle approaches 0 from the positive side and π from the negative side). Similarly, setting $\theta \to 0$ or $\theta \to \pi$ in (1) gives $r \to 0$. Therefore we can set

$$\theta \in [0, \tfrac{\pi}{2}) \cup (\tfrac{\pi}{2}, \pi).$$

Of course, since sine and cosine are periodic this is not the only possible domain for θ. It is just one possibility that covers all points of the parabola. A more elegant choice would be for example

$$\theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

This is effectively the same thing as introducing the substitution $x = \tan\theta$, which is a bijection for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The argument can also be represented purely geometrically, without the dot product (see Figure). The important thing is that we cover all points and avoid dividing by zero. An animated illustration is available here.

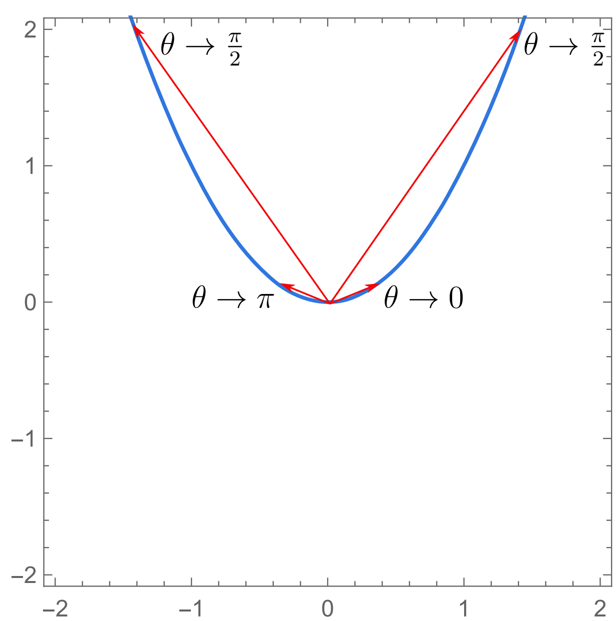

Figure 1: Various values of θ for points on the parabola $y = x^2$

b) We can rewrite the equation of the surface as

$$r^2(\sin^2\theta + 2\cos^2\theta) - z^2 = 1.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$ this is equivalent to

$$2x^2 + y^2 - z^2 = 1,$$

which describes an elliptic hyperboloid of one sheet.

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Problem 6 (1.5 points)

Let C be the curve parametrized by

$$\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(2t)\,\mathbf{j},$$

with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

a) Show that C is a closed curve without self-intersections (no point is visited twice by $\mathbf{r}(t)$ on the given domain, except the initial point).

b) Compute the following two integrals and verify that they are equal:

i) $\displaystyle\int_C x\,\mathrm{d}y,$ ii) $\displaystyle -\int_C y\,\mathrm{d}x.$

The two integrals equal the area of the region enclosed by C.

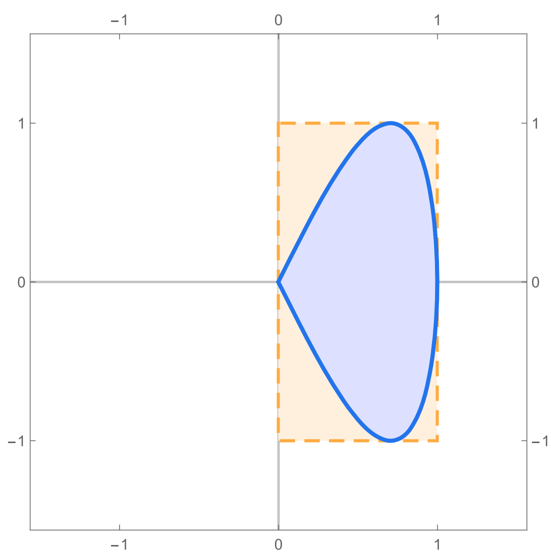

Figure 1: The region enclosed by C and a rectangle of area 2 for comparison

Solution

a) Suppose that $\mathbf{r}(s) = \mathbf{r}(t)$. This implies (after applying the double-angle formula for sine)

$$\cos(t) = \cos(s) \quad \wedge \quad \sin(t)\cos(t) = \sin(s)\cos(s).$$

Clearly $t = \frac{\pi}{2}$ and $s = -\frac{\pi}{2}$ give the same point. For other values we have $\cos(t) \neq 0$ so we can divide the second equation and obtain

$$\cos(t) = \cos(s) \quad \wedge \quad \sin(t) = \sin(s).$$

The only way this is possible for $s, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is if $t = s$ so the curve is indeed closed and without self-intersections.

b) We have $\mathbf{r}'(t) = -\sin(t)\,\mathbf{i} + 2\cos(2t)\,\mathbf{j}$. Therefore

$$\begin{aligned}
\int_C x\,\mathrm{d}y &= \int_{-\pi/2}^{\pi/2} x(t)\,y'(t)\,\mathrm{d}t = \int_{-\pi/2}^{\pi/2} \cos(t)\,2\cos(2t)\,\mathrm{d}t \\
&= 2\int_{-\pi/2}^{\pi/2} \cos(t)\,(1 - 2\sin^2(t))\,\mathrm{d}t \qquad (u = \sin(t)) \\
&= 2\int_{-1}^{1} (1 - 2u^2)\,\mathrm{d}u = 2\left(u - \tfrac{2}{3}u^3\right)\Big]_{-1}^{1} = 4\left(1 - \tfrac{2}{3}\right) = \frac{4}{3}.
\end{aligned}$$

$$\begin{aligned}
-\int_C y\,\mathrm{d}x &= -\int_{-\pi/2}^{\pi/2} y(t)\,x'(t)\,\mathrm{d}t = \int_{-\pi/2}^{\pi/2} \sin(t)\,\sin(2t)\,\mathrm{d}t \\
&= 2\int_{-\pi/2}^{\pi/2} \sin^2(t)\,\cos(t)\,\mathrm{d}t \qquad (u = \sin(t)) \\
&= 2\int_{-1}^{1} u^2\,\mathrm{d}u = \tfrac{2}{3}u^3\Big]_{-1}^{1} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.
\end{aligned}$$

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Problem 7 (1.5 points)

Given are two vector fields,

$$\mathbf{F} = (z^2\cos(x) + 2x\sin(y))\,\mathbf{i} + x^2\cos(y)\,\mathbf{j} + 2z\sin(x)\,\mathbf{k},$$

$$\mathbf{G} = (z^2\cos(x) + 2x\cos(y))\,\mathbf{i} + x^2\sin(y)\,\mathbf{j} + 2z\sin(x)\,\mathbf{k}.$$

a) Determine which of the two vector fields has greater divergence at the point $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$.

b) One of the two vector fields is conservative and the other is not. Determine which is which.

c) For the vector field that is conservative, compute the potential function whose value at $(0,0,0)$ is zero.

d) Compute the integrals of $\mathbf{F}$ and $\mathbf{G}$ along the line segment connecting the points $(0,0,0)$ and $(1,0,1)$.

Hint: Vector fields $\mathbf{F}$ and $\mathbf{G}$ look very similar so it might be worth considering the vector field $\mathbf{F} - \mathbf{G}$ to save on computations.

Solution

a) We have

$$\mathbf{F} - \mathbf{G} = (\sin(y) - \cos(y))(2x\,\mathbf{i} - x^2\,\mathbf{j}).$$

Since differentiation is a linear operation we have

$$\begin{aligned}
\mathrm{div}\mathbf{F} - \mathrm{div}\mathbf{G} &= \mathrm{div}(\mathbf{F} - \mathbf{G}) \\
&= (\sin(y) - \cos(y))\tfrac{\partial}{\partial x}(2x) - x^2\tfrac{\partial}{\partial y}(\sin(y) - \cos(y)) \\
&= 2(\sin(y) - \cos(y)) - x^2(\sin(y) + \cos(y)).
\end{aligned}$$

For the given point we have

$$\operatorname{div}(\mathbf{F}-\mathbf{G})(\tfrac{\pi}{2},\tfrac{\pi}{2},\tfrac{\pi}{2}) = 2 - \frac{\pi^2}{4} < 0$$

and therefore $\mathbf{G}$ has greater divergence at $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

b) Write $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{j}$ and $\mathbf{G} = G_1\,\mathbf{i} + G_2\,\mathbf{j} + G_3\,\mathbf{k}$. Then

$$\frac{\partial G_1}{\partial y} = -2x\sin(y) = -\frac{\partial G_2}{\partial x}$$

and so $\operatorname{curl}\mathbf{G} \neq \mathbf{0}$, which implies $\mathbf{G}$ is not conservative. We verify that $\mathbf{F}$ is conservative by computing:

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= 2x\cos(y) = \frac{\partial F_2}{\partial x},\\ \frac{\partial F_1}{\partial z} &= 2z\cos(x) = \frac{\partial F_3}{\partial x},\\ \frac{\partial F_2}{\partial z} &= 0 = \frac{\partial F_3}{\partial y}.\end{aligned}$$

Therefore $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and since the domain of $\mathbf{F}$ is $\mathbb{R}^3$ we conclude that $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla f$.

c) Integrating F_i with respect to the i-th variable gives

$$\begin{aligned}f(x,y,z) &= \int F_1(x,y,z)\,\mathrm{d}x = z^2\sin(x) + x^2\sin(y) + C_1(y,z),\\ f(x,y,z) &= \int F_2(x,y,z)\,\mathrm{d}y = x^2\sin(y) + C_2(x,z),\\ f(x,y,z) &= \int F_3(x,y,z)\,\mathrm{d}z = z^2\sin(x) + C_3(x,y).\end{aligned}$$

Combining the three and the constraint $f(\mathbf{0}) = 0$ gives

$$f(x,y,z) = z^2\sin(x) + x^2\sin(y).$$

d) The path C can be parametrized by $\mathbf{r}(t) = (t,0,t)$ with $t \in [0,1]$. By the fundamental theorem for path integrals, we have

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C \nabla f\cdot\mathrm{d}\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = \sin(1).$$

Since $\mathbf{r}'(t) = (1,0,1)$ we have

$$\int_C (\mathbf{F}-\mathbf{G})\cdot\mathrm{d}\mathbf{r} = \int_0^1 (\mathbf{F}(\mathbf{r}(t)) - \mathbf{G}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_0^1 -2t\,\mathrm{d}t = -1.$$

Hence $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 1 + \int_C \mathbf{G}\cdot\mathrm{d}\mathbf{r} = 1 + \sin(1)$.