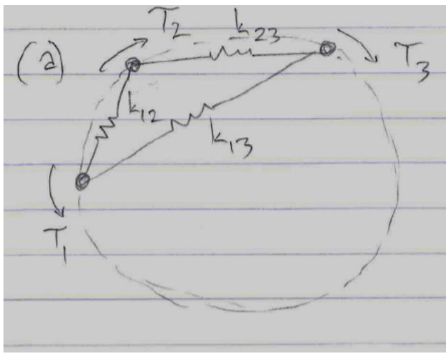
Mock exam solutions

Disclaimer: The following is not an exhaustive list, i.e. you will probably be asked to conceptualize and analyze different physical (or virtual) systems from the ones in the list, or even combinations of such systems. **The full course content is fair game for the exams** (ref. “How should I study for these exams?”)

Note that information will be provided to you without schematics (where possible) in these exams.

- Three particles are constrained to move on a unit circle without colliding. Each particle of mass M_i is driven by torque T_i, and experiences a viscous damping force that opposes its motion and scales with a damping coefficient D_i. Each pair of particles is interconnected with a spring of stiffness k_{ij}.
  a) Draw the free-body diagram of the system defined above and include all necessary information.

(a)



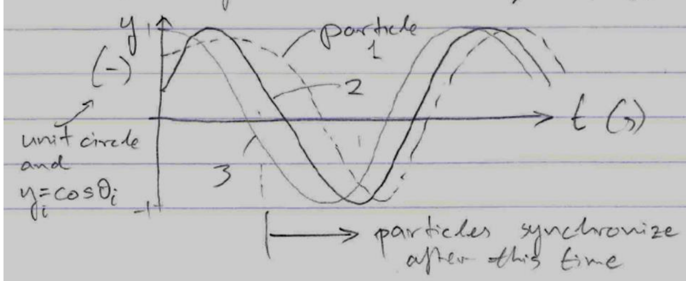
Angles θ_i
Frequencies $\dot{\theta}_i$
Inertial coefficients (masses) M_i
Damping coefficients D_i
Stiffness coefficients $k_{ij} = k_{ji} > 0$ (for pairs of particles)
Torques $T_i \in \mathbb{R}$

  b) Describe qualitatively what you expect will be the behavior of the system as a function of the magnitude of the masses, the damping coefficients and spring stiffness values.

(b) Qualitative description:
If the masses are very large, inertial forces will dominate the behavior, especially if viscous damping is small in comparison; hence, synchronization would be difficult to achieve. In the opposite case of small masses and high viscous damping, synchronization is more likely to occur depending on the coupling strength, which scales with the spring stiffness between each pair of particles.

  c) What do you expect will be this behavior if the natural frequencies of the system are numerically close to each other versus when they are strongly dissimilar? Use plots of the kinematics (displacement and velocity time histories) and dynamics (force and other derived quantity time histories) as necessary to support your arguments.

(c) For uniform damping $D_i = D$, the natural frequencies are $\omega_i = T_i/D$. If the numerical values of ω_i are close to each other (i.e. a homogeneous network), the masses will synchronize for sufficiently strong coupling, provided that inertial forces do not dominate (see part b). It is sufficient to demonstrate this by plotting, for example, the vertical displacement of the particles (or any other kinematic quantity).



In the opposite case (of no synchronization) the particles continue on their original trajectories

d) Derive the equation of motion of each particle and present the equation describing the system using indicial notation (i.e. using M_i, D_i, T_i, etc.). List all relevant assumptions used in your model.

(d) $M_i \ddot{\theta}_i + D_i \dot{\theta}_i = T_i - \sum_{j=1}^{3} k_{ij} \sin(\theta_i - \theta_j)$ for $i \in [1, 2, 3]$

All coefficients are assumed to be positive ($M_i > 0$, $D_i > 0$, $k_{ij} = k_{ji} > 0$) and the signs of the torques T_i represent their direction (clockwise or counterclockwise). Also, the elastic restoring torque for each pair of particles is $k_{ij} \sin(\theta_i - \theta_j)$. Finally, the viscous damping forces $D_i \dot{\theta}_i$ oppose the direction of motion, while no collisions occur between the particles.

e) Simplify the equation of motion assuming uniformly high viscous damping $D = D_i$ and small masses, and express the equation in terms of the particles' natural frequency ω_i instead of their applied torque T_i.

(e) For $D_i = D$ and small masses, the ratio $M_i/D \approx 0$ so that the inertial forces $\left(\frac{M_i}{D}\right)\ddot{\theta}_i \approx 0$ when we divide through with D.

$$\therefore \quad \dot{\theta}_i = \frac{T_i}{D} - \sum_{j=1}^{3} \frac{k_{ij}}{D} \sin(\theta_i - \theta_j)$$

and let $\omega_i = \frac{T_i}{D}$ and $a_{ij} = \frac{k_{ij}}{D}$.

$$\Rightarrow \quad \dot{\theta}_i = \omega_i - \sum_{j=1}^{3} a_{ij} \sin(\theta_i - \theta_j)$$

where ω_i are the natural frequencies and a_{ij} are the coupling strengths.

f) Write pseudo-code to describe how your model can be solved numerically and briefly discuss expected numerical issues (type of solver, convergence issues, etc.).

(f) Pseudocode (refer to lecture notes and assignments):

Define constants...
Initialize vectors...
Run over defined number of iterations
for j = 1 : number of iterations
    Calculate $\theta(i,j)$ using RK45 (ode45)
    ⋮
    Calculate x and y (Cartesian) coordinates
    Plot results on circular trajectory
end

} You are expected to go into as much detail as needed here...

g) How could you model collisions between these particles? Use mathematical expressions to explain your answer.

(g) Simplest way is to activate reaction velocities upon collision using, for example, a coefficient of restitution e. Alternatively, a reaction force could be defined (as we did in the bouncing pendulum case). Let's look at the first option here...

Cons. of momentum: $M_i u_i + M_j u_j = M_i v_i + M_j v_j$

Cons. of kinetic energy: $\frac{1}{2} M_i u_i^2 + \frac{1}{2} M_j u_j^2 = \frac{1}{2} M_i v_i^2 + \frac{1}{2} M_j v_j^2$

where u_i and u_j are initial velocities and v_i and v_j are the velocities after the collision of particles i and j.

After some algebra, and by introducing the coefficient of restitution $e = \left|\frac{v_i - v_j}{u_i - u_j}\right|$; for perfectly elastic collisions we can assume that $e = 1$ (good enough for our purposes).

$\therefore$

$$v_i = \frac{e M_j (u_j - u_i) + M_i u_i + M_j u_j}{M_i + M_j}$$

$$v_j = \frac{e M_i (u_i - u_j) + M_i u_i + M_j u_j}{M_i + M_j}$$

Rotational velocities $\dot{\theta}$ are related to tangential velocities u or v as $u = \overset{1}{r}\dot{\theta} \Rightarrow u = \dot{\theta}$

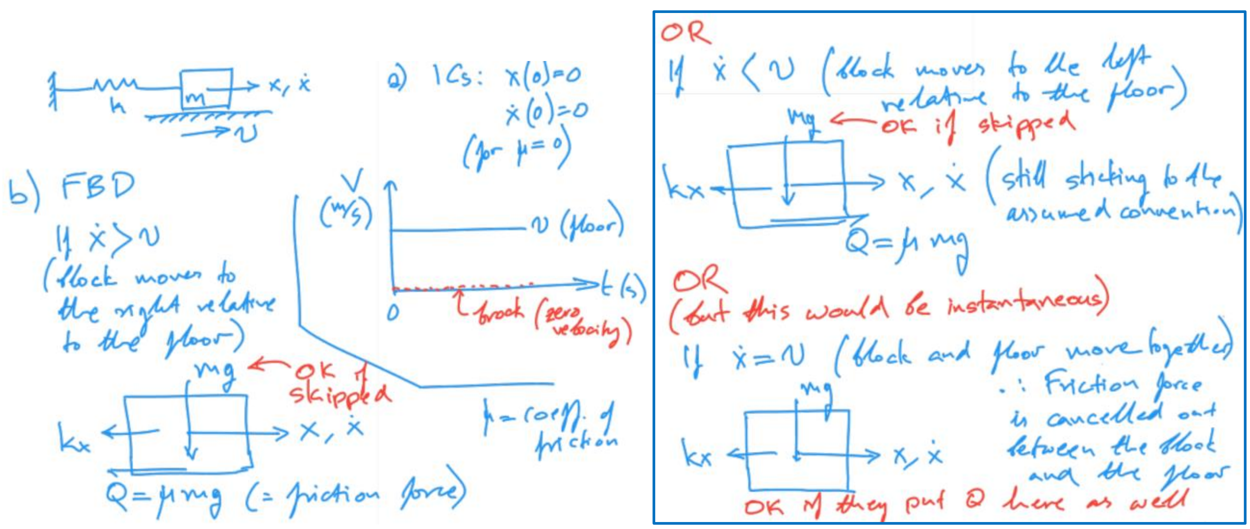
For a time index m (i.e. m = current time and $m+1$ = time after one time step Δt), we could write this as follows:

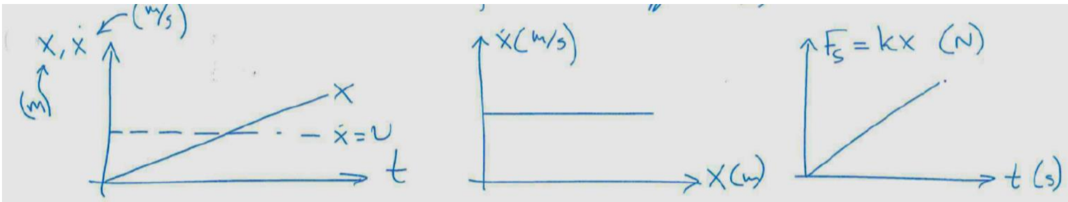
$$\dot{\vartheta}_i^{k+1} = \left[e M_j (\dot{\vartheta}_j - \dot{\vartheta}_i) + M_i \dot{\vartheta}_i + M_j \dot{\vartheta}_j \right] / (M_i + M_j)$$
$$\dot{\vartheta}_j^{k+1} = \left[e M_i (\dot{\vartheta}_i - \dot{\vartheta}_j) + M_i \dot{\vartheta}_i + M_j \dot{\vartheta}_j \right] / (M_i + M_j)$$

and collision would occur if $|\vartheta_i^{k+1} - \vartheta_j^{k+1}| < |\vartheta_i^k - \vartheta_j^k|$ meaning that the distance between a pair of particles becomes smaller (i.e. they approach each other) plus if $|\vartheta_i^{k+1} - \vartheta_j^{k+1}| < \delta$ where δ is some cutoff distance.

Note: the last part of this example is more difficult than what you would be asked to solve in an exam; however, drawing from your knowledge of high-school physics is to be expected.

- A block of mass m rests at its equilibrium position while being connected to a spring of stiffness k that is fixed to a rigid wall installed at the left of the block. The floor on which the block rests moves to the right with velocity v at time $t = 0\ s$.
  a) Assume there is no friction between the block and the floor. Write down the initial conditions of the system and plot the velocities of the block and the floor as functions of time.
  b) Let the friction force be nonzero and proportional to the weight of the block by a constant coefficient μ. Sketch the free-body diagram of the system. Plot the kinematics (i.e. the displacement and velocity as a function of time), the phase plot, as well as the time history of the spring force under the assumption that friction is essentially infinite (i.e. $\mu \rightarrow \infty$).





c) Using cutting-edge technology, assume we can turn friction on and off at will (i.e. $\mu \to \infty$ for the "on" condition and $\mu = 0$ for the "off" condition). Initially, friction is "on" until we turn it "off" at $t = 0\ s$. If the stiffness of the spring is $16\pi^2\ N/m$ and the mass of the block is 1 kg, how many times does the center of the block cross the equilibrium position during the first 10 seconds? Plot the kinematics and phase plot of the system.

c) $k = 16\pi^2\ N/m$

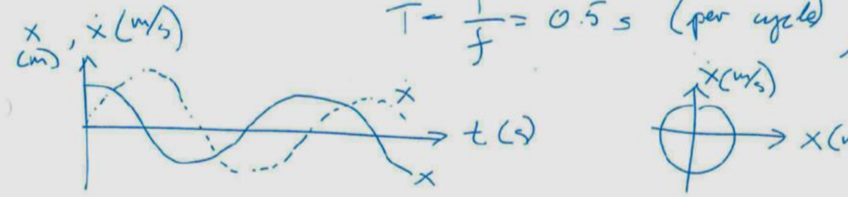
$m = 1\ kg$

$\omega_n = \sqrt{\frac{k}{m}} = 4\pi\ rad/s$

$f = \frac{\omega_n}{2\pi} = 2\ Hz\ (= 2\ cycles/s)$

$T = \frac{1}{f} = 0.5\ s$ (per cycle)

2 crossings per cycle ∴ 2 cross. $\cdot \frac{10 s}{0.5 s}$ = 40 cross.



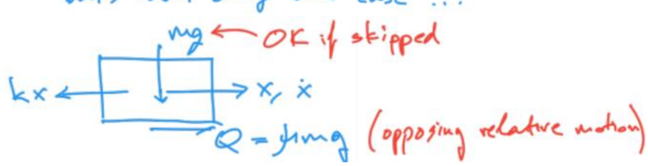
d) Using free-body diagrams, sketch two of the possible states of the system described in part c): positive displacement (block positioned to the right of the equilibrium position) and negative displacement (block positioned to the left of equilibrium). Include and define the internal and external forces acting on the block in each state.

e) Given Newton's second law (the acceleration of an object depends directly upon the net force acting on the object, and inversely upon the mass of the object) derive the equation of motion for the system of parts c) and d).

d) Positive displacement $x > 0$

if $\dot{x} = v$ OR $\dot{x} > v$ OR $\dot{x} < v$ as per part b). Be lenient

Negative displacement $x < 0$. Since the floor always moves to the right, this has only one case ...



e) $\Sigma F = m\ddot{x} \Rightarrow -kx - \mu mg = m\ddot{x}$ ← could also be + also in f

$\Rightarrow m\ddot{x} + kx = \mu mg$ OR $m\ddot{x} + kx = -\mu mg$

(A function like $sgn(\dot{x})$ or $sgn(\dot{x} - v)$, etc. could be used to switch the force, but I don't expect students to know that.)

f) Write pseudocode to show how the behavior of the system described in part e) can be approximated numerically by solving a system of two first-order ODEs (e.g. by calling "ode45"). Make sure to write down explicitly the system of equations to be solved.

f) Let $\dot{x} = y$ so that $\ddot{x} = \dot{y}$. Then $m\dot{y} + kx = \mu mg$

$$\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ \mu mg \end{Bmatrix}$$

Extra pts: use sign(x) to change the direction of the friction force (also in part e).

$$m\ddot{x} + kx = \mathrm{sgn}(x)\,\mu mg$$

Pseudocode: constants
initializations
ode45
⋮

- The Bernoulli equation for the steady-state flow of an incompressible fluid (expressed in terms of the total head) has the following form:

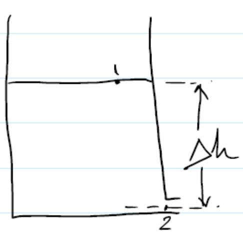
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}.$$

Consider the case of a reservoir containing water ($\rho = 998\ kg/m^3$) draining via a small horizontal pipe of diameter d situated at a distance Δh below the waterline (initial water volume is V).

a) Derive an equation for the volumetric flow-rate (Q) through this pipe as a function of the fluid height. Assume that the flow is steady and friction does not play a role.

$$\frac{P}{\rho g} + \frac{v^2}{2g} + z = H = \text{constant}$$

p = pressure
ρ = density
v = velocity
g = grav. const.
z = height
H = total head



(a)

$$\frac{\overset{p_{atm}}{p_1}}{\rho g} + \frac{\overset{0}{\cancel{v_1^2}}}{2g} + z_1 = \frac{\overset{p_{atm}}{p_2}}{\rho g} + \frac{\overset{v}{v_2^2}}{2g} + z_2$$

$$\Rightarrow \cancel{\frac{p_{atm}}{\rho g}} + z_1 = \cancel{\frac{p_{atm}}{\rho g}} + \frac{v^2}{2g} + z_2 \Rightarrow$$

$$v^2 = 2g(\underbrace{z_1 - z_2}_{\Delta h}) = 2g\Delta h$$

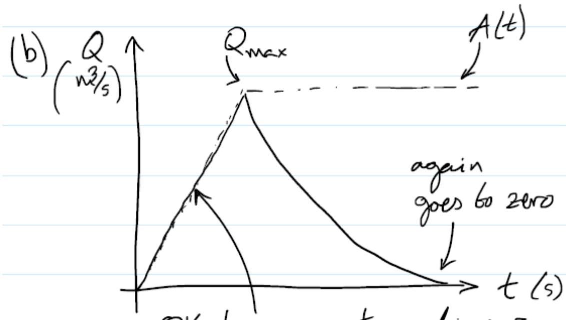
Assume for (a), A = const.

$\therefore\ Q = Av \Rightarrow \boxed{Q = A\sqrt{2g\Delta h}}$

b) Assuming that we fit a valve onto the outflow pipe, sketch $Q(t)$ when the valve opening goes from fully closed to fully open over an amount of time Δt, and remains fully open thereafter. Write down the corresponding pseudocode needed to calculate the flow-rate.

c) Sketch $Q(t)$ for a liquid solution with a higher density than water.

(b)



OK to represent as linear
(note: Δh decreases with flow)

Pseudocode:

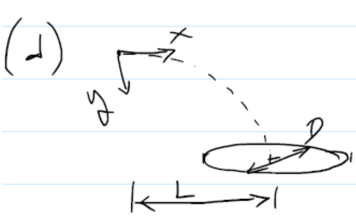
```
t = linspace(0, tmax, N);
for i=2: length(N)
    if A(i) < Amax
        A(i) = Aslope t;
    else
        A(i) = Amax;
    end
    Q(i) = A(i)*sqrt(2*g* Δh(i));
    V(i) = V(i-1) - Q(i) * dt;
end
```

(c) $Q(t)$ is independent of density; hence, the graph will be the same as in (b).

Now, consider that we want to collect the outflow of the original reservoir into a second reservoir having an inlet diameter D and whose center (of the inlet), due to design considerations, sits at distance L away from the end of the outflow pipe of the original reservoir.

d) If the instantaneous vertical position of a particle of this outflow jet is given by $y = gt^2/2$, where t is time, derive an expression for the height difference between the outflow pipe center and the inlet of the second reservoir. Assume that the horizontal velocity component is constant.

(d)



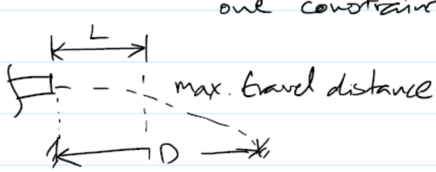
$y = \frac{gt^2}{2}$ and $x = vt$ given $x = L$

$\Rightarrow t = \frac{x}{v} = \frac{L}{v}$

$\therefore \boxed{y = \frac{g}{2}\left(\frac{L}{v}\right)^2}$

e) What are acceptable dimensions for the second reservoir's inlet to avoid spills during operation? Hint: how far does the water jet travel as a function of the available hydraulic head?

(e) Assumption: as $v \to 0$, the liquid will flow (drip) into the reservoir below; hence, $D \geq 2L$ is the one constraint for the case of little flow



But, what should the limit for how big D should be?

$v = \sqrt{2g\,\Delta h}$ and $x = vt = t\sqrt{2g\,\Delta h}$

$\therefore x_{max} = t\sqrt{2g\,\Delta h_{max}}$ and $D \geq x_{max}$

So, $\boxed{D \geq 2L \text{ and } D \geq t\sqrt{2g\Delta h_{max}}\ .}$

Vertical distance limit?

$y = \frac{g}{2}\left(\frac{L}{v}\right)^2$ and $v_{max} = \sqrt{2g\Delta h_{max}}$

$\therefore y_{max} = \frac{g}{2}\frac{L^2}{2g\Delta h_{max}} \Rightarrow \boxed{y_{max} = \frac{L^2}{4\Delta h_{max}}}$

Do not deduct points if this is missing; if this is included, give 5 bonus points

Finally, consider that the original reservoir filled with solution A flows into the second reservoir that contains solution B. The two species together form the product P with stoichiometry 1:1, and the reaction is 1st order in both species. The rate of A (mass per second) entering tank 2 depends on the flow-rate and concentration of the solution.

f) Write down the differential equations that describe the evolution of all three compounds in terms of their masses (so, not concentration!)

g) Write pseudocode on how to solve these equations. For the purpose of simplification and convenience, you don't need to capture the change of liquid height in tank 1 and its effect on flow $Q(t)$.

(f) $\frac{dm_A}{dt} = Q(t)\, C_A - k_1 C_A C_B V_2$ where $Q(t) = A\sqrt{2g\Delta h(t)}$

$\frac{dm_B}{dt} = -k_1 C_B C_A V_2$

$\frac{dm_P}{dt} = k_1 C_A C_B V_2$

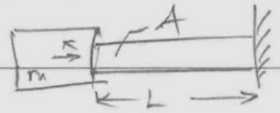
(g) Pseudocode (refer to assignment rubric)

h) Suppose that we can choose between two vessel designs for reservoir 1 that have the same volume V: variant 1, which is wider and shorter, or variant 2 with a smaller diameter and greater height (the cross sectional area of the horizontal outlet pipe would remain the same in both cases). Which would you choose to speed up the process, considering the chemical reaction taking place in reservoir 2? (Assume that the reaction is limited by the supply of species A.)

(h) The rate of A entering reservoir 2 depends on the flowrate, which is a fcn of Δh in reservoir 1. Hence, variant 2 will speed up the reaction because of its greater height $\rightarrow$ greater Δh.

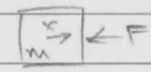
Dynamics of Engineering Systems

- Consider a rod of length L and cross-sectional area A that is fixed (horizontally) to a rigid wall on one side; the temperature of the system (rod, wall and their environment) is 60°C. A rigid mass m impacts the free side of the rod which then deforms by an amount x.
  a) Draw the free-body diagram of the mass and write the equation of motion for the mass as it compresses the rod (consider the problem in 1D, i.e. only horizontal movement).



Viscoelastic rod of cross-section A and length L impacted by mass m

(a) $m\ddot{x} = -F \Rightarrow \boxed{m\ddot{x} + F = 0}$ FBD:



  b) Assume that the free end of the rod is subjected to cyclic loading with $x = x_0 e^{i\omega t}$ (i.e. replace the displacement in the equation of section (a) using the new expression given for the displacement). Rewrite the equation of motion in terms of stress and strain and solve for their ratio (σ/ε).

(b) Let $x = x_0 e^{i\omega t}$. Then, $\dot{x} = i\omega x_0 e^{i\omega t}$ and

(Note: $\sigma = \frac{F}{A} \Rightarrow F = \sigma A$) $\ddot{x} = i^2\omega^2 x_0 e^{i\omega t} = -\omega^2 x_0 e^{i\omega t}$

Then,

$$-\omega^2 m x_0 e^{i\omega t} + \sigma A = 0 \Rightarrow \sigma = \frac{\omega^2 m x_0}{A} e^{i\omega t}$$

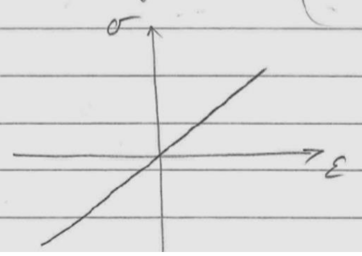
Strain is $\varepsilon = \frac{x}{L} = \frac{x_0}{L} e^{i\omega t} \Rightarrow x_0 = \varepsilon L / e^{i\omega t}$

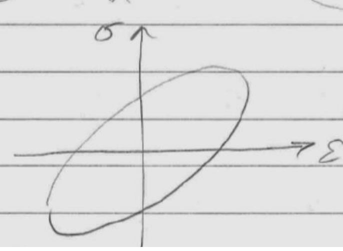
Plugging into the EOM yields: $\sigma = \frac{\omega^2 m}{A} \frac{\varepsilon L}{e^{i\omega t}} e^{i\omega t}$

such that $\boxed{\frac{\sigma}{\varepsilon} = \frac{\omega^2 m L}{A}}$

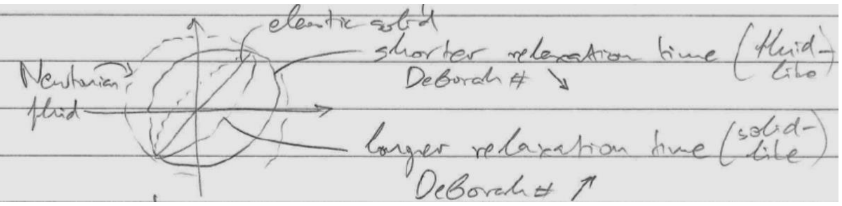
  c) Plot the stress-strain response of the rod material for the following cases of cyclic loading: purely elastic material; and, viscoelastic (Maxwell) material.

(c) If purely elastic, (then $\frac{\sigma}{\varepsilon} = \frac{\omega^2 m L}{A} = E \Rightarrow$ ($\omega^2 = \frac{EA}{mL}$ aside))





Bonus:



d) Assuming that the rod material is viscoelastic, set the ratio found in section (b) equal to the complex modulus of a Maxwell material given by $\left[\frac{1}{E} - \frac{i}{\eta\omega}\right]^{-1}$, where E is the elasticity and η is the viscosity, and solve the resulting equation for ω. Using your understanding of the physics of such oscillatory systems, discuss the conditions for which the system exhibits damped vibration or aperiodic damping.

(d) $\frac{\omega^2 mL}{A} = \left[\frac{1}{E} - \frac{i}{\eta\omega}\right]^{-1} \Rightarrow \frac{A}{\omega^2 mL} = \frac{1}{E} - \frac{i}{\eta\omega}$ $\qquad \omega_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \boxed{\left(\frac{1}{E}\right)\omega^2 - \left(\frac{i}{\eta}\right)\omega - \frac{A}{mL} = 0}$ Use quadratic formula with $a = 1/E$, $b = -i/\eta$ and $c = -A/mL$

to get $\boxed{\omega_{1/2} = i\left(\frac{E}{2\eta}\right) \pm \frac{E}{2}\sqrt{\frac{4A}{mLE} - \frac{1}{\eta^2}}}$

Imaginary root → aperiodic damping

This occurs when $\frac{4A}{mLE} - \frac{1}{\eta^2} < 0 \Rightarrow \boxed{\frac{E}{\eta^2} > \frac{4A}{mL}}$ Derivation of conditions ⇓ full marks

Real root → damped vibration when $\frac{E}{\eta^2} < \frac{4A}{mL}$

small m, large E

The rigid mass impacting the viscoelastic rod has the geometry of a sphere of radius R that is initially at a temperature of 25°C. Following impact, the spherical mass sticks to the viscoelastic rod so that heat transfer occurs in the system due to conduction only (in 1D).

e) What are the initial and boundary conditions for the spherical mass?

f) Sketch the 1D temperature distribution along the sphere's center (from $-R$ to R) as a function of time for 5 time increments (the last one for time tending to infinity). For simplicity, assume that the temperature profile in the **rod** remains constant, even after impact.

g) The equation for heat conduction is given by $\frac{\partial T}{\partial t} = \alpha\nabla^2 T$. Write down the equations approximating the behavior of this system in 1D along the centerline of the sphere ($-R \leq x \leq R$) using the second order central difference equation for space and the forward difference equation for time.

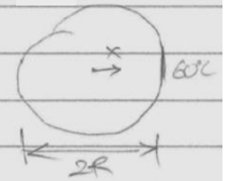
e) ICs (for the mass): $T(x=R, t=0) = 60°C$

$T(x<R, t=0) = 25°C$

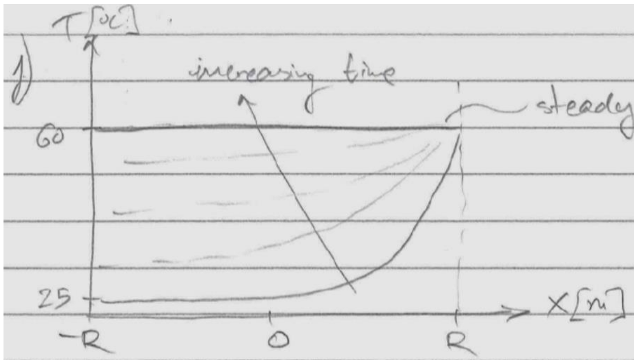
BCs (— " —): $T(x=R, t) = 60°C$

$\left.\frac{\partial T}{\partial x}\right|_{x=-R} = 0$

extra answer



f)



Getting the inverse behavior (as in last year's exam) results in 0 points.

g) $\frac{\partial T}{\partial t} \approx \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$

$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i-1}^{j} + 2T_i^{j} - T_{i+1}^{j}}{\Delta x^2}$

$$T_i^{j+1} = T_i^j + \alpha \Delta t \left(\frac{T_{i-1}^{j} + 2T_i^{j} - T_{i+1}^{j}}{\Delta x^2} \right)$$

8 + 8 + 4 = 20