Question 1 (35pt) Consider the function

$$f(x_1, x_2, x_3) = x_1\big(ax_1 + 6x_2\big) + b(x_2 - 3)^2 - cx_3^2$$

for some scalars a, b, and c.

Part (i): Determine the condition(s) on a, b, and c, such that the function f is strictly convex.

Part (ii): Set $a = 3(s+1)$ and $b = \frac{5}{s+1}$, where s is the last digit of your student number. Compute the minimum of f.

Part (iii): Consider the function f again and set $a = 18$, $b = 1$ and $c = 0$. This results in the cost function

$$f(x_1, x_2) = x_1\big(18x_1 + 6x_2\big) + (x_2 - 3)^2.$$

We add the equality constraint

$$30x_1 + 4x_2 = m$$

where m is a positive number. Determine the value of m such that the optimal value of x_1 becomes equal to 1, that is $x_1^* = 1$. (Include also the optimal value x_2^* and the Lagrangian multiplier λ^* in your answer.)

Note: If you were not able to find m, write your solutions for the rest of this question in terms of m.

Part (iv): Write a primal-dual algorithm for the constrained optimization problem in part (iii). Moreover, prove that the equilibrium of your algorithm is asymptotically stable.

Part (v): Write a primal-dual algorithm for the constrained optimization problem in part (iii), but this time with an augmented Lagrangian function.

Question 2 (16pt) Ohm's law states that the voltage across a (linear) resistor is proportional to the current passing through the resistor, namely $V = RI$, where V is the voltage, I is the current, and R is the resistance. For a given resistor, we have performed 3 experiments and collected the data as

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}, \qquad \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}, \qquad \begin{pmatrix} V_3 \\ I_3 \end{pmatrix}$$

where V_i is the measured voltage and I_i is the measured current in the ith experiment, for each $i = 1, 2, 3$. Assuming that the resistor approximately satisfies the Ohm's law, we would like to compute its resistance R in terms of the collected data using the least square method. For this purpose, we have 3 processors to use where processor i has access to the values V_i and I_i.

Part (i): Formulate this least square problem as a distributed optimization problem.

Part (ii): For this optimization problem, write down explicitly a (distributed) primal-dual algorithm with Laplacian matrix constraints corresponding to a communication graph with two links.

Question 3 (24pt) Consider a data network with three sources and one link, where all sources use that link. The utility function for each source is given by $x_i(x_i - \gamma_i)$ where x_i is the rate of sending data at the ith source, and $\gamma_i > 0$ for each $i = 1, 2, 3$. The nominal capacity of the link is given by c.

Part (i) The aim of internet congestion control is to maximize the sum of the utility functions while setting the aggregate flow at the link equal to c. Write the routing matrix R and the corresponding optimization problem. Convert this maximization to a minimization problem, and then design a *dual ascent* dynamical algorithm.

Part (ii) Consider again the optimization problem in part (i). We split c into three parts, and set

$$c = c_1 + c_2 + c_3,$$

where $c_i > 0$ for each $i = 1, 2, 3$. Design a distributed primal-dual algorithms using incidence matrix constraints for this optimization problem, in which source i has access to c_i but not to c_j with $j \neq i$.
Hint: View the optimization problem as an optimal resource allocation problem, where c_i plays the role of the demand at node i.