Calculus 2 (IEM)

Midterm Exam II

12 March 2021

university of groningen

Problem 1

Given is a surface in $\mathbb{R}^3$ whose equation in cylindrical coordinates is

$$z = r\cos 3\theta.$$

a) Convert the given equation from cylindrical coordinates (r, θ, z) to Cartesian coordinates (x, y, z) to define a function $z = f(x, y)$.

b) Show that the function f defined in a) is continuous at $(0, 0)$.

c) Determine whether f is differentiable at $(0, 0)$.

Solution

Cylindrical coordinates are given by $(x, y, z) = (r\cos\theta, r\sin\theta, z)$.

a) Using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $r^2 = x^2 + y^2$ we get

$$z = r(4\cos^3\theta - 3\cos\theta) = \frac{4r^3\cos^3\theta}{r^2} - 3r\cos\theta = \frac{4x^3}{x^2+y^2} - 3x.$$

Therefore

$$f(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

b) Since $f(0,0) = 0$ and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r\cos 3\theta = 0,$$

we conclude that f is continuous at zero.

c) We first compute the partial derivatives at $(0, 0)$:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{x}{x} = 1,$$
$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{0}{y} = 0.$$

Therefore the linearization of $f(x,y)$ at $(0,0)$ is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x.$$

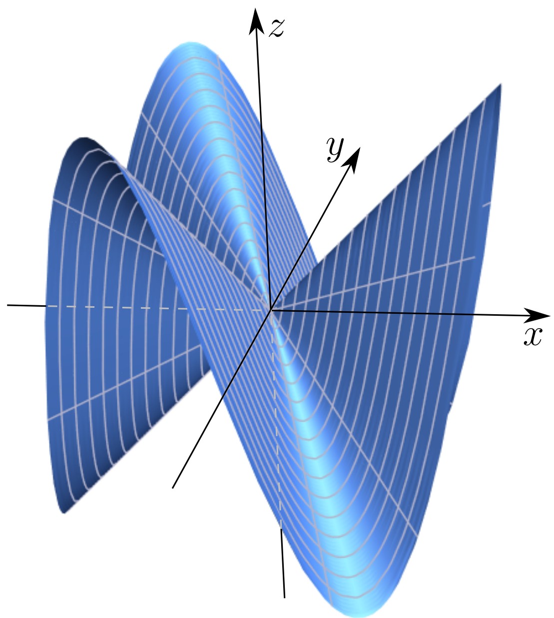
By definition, f is differentiable at $(0,0)$ if

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - L(\Delta x,\Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

Since

$$\lim_{r\to 0^+} \frac{r\cos 3\theta - r\cos\theta}{r} = \lim_{r\to 0^+} (\cos 3\theta - \cos\theta)$$

does not exist (it depends on θ), it cannot be zero and we conclude that f is not differentiable at $(0,0)$.



The graph of f does not look flat near $(0,0)$ – there are infinitely many directions for the tangent at this point that are not all in the same plane.

Remark: Another way to compute the partial derivatives is as follows. On the positive half of the x-axis we have $\theta = 0$ and on the positive half of the y-axis we have $\theta = \frac{\pi}{2}$. Thus

$$f_x(0,0) = \lim_{r\to 0} \frac{z(r,0) - z(0,0)}{r\cos 0 - 0} = \lim_{r\to 0} \frac{r\cos(3\cdot 0)}{r} = \lim_{r\to 0} \cos 0 = 1,$$

$$f_y(0,0) = \lim_{r\to 0} \frac{z(r,\frac{\pi}{2}) - z(0,0)}{r\sin\frac{\pi}{2} - 0} = \lim_{r\to 0} \frac{r\cos(3\cdot\frac{\pi}{2})}{r} = \lim_{r\to 0} \cos\frac{3\pi}{2} = 0.$$

Note that $r \to 0$ includes both positive and negative values of r; the limits must match from the left and from the right (so $\theta = 0$ also covers $\theta = \pi$, while $\theta = \frac{\pi}{2}$ also covers $\theta = -\frac{\pi}{2}$).

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Problem 2

Consider the function $f(x, y) = \sin x \cdot \sin y$ and let $(x_0, y_0) = (\frac{\pi}{2}, \frac{\pi}{3})$.

a) Determine the partial derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} at (x_0, y_0).

b) Compute and simplify the function

$$Q(x, y) = f(x_0, y_0) + (A(x, y) + \mathbf{b}) \bullet (x, y),$$

where $\mathbf{b} = (f_x, f_y)$ and A is the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix},$$

with the derivatives evaluated at (x_0, y_0).

c) Classify the quadric surface $z = Q(x, y)$.

Solution

a) Differentiating with respect to x and y respectively gives

$$\frac{\partial f}{\partial x} = \cos x \cdot \sin y, \qquad \frac{\partial f}{\partial y} = \cos y \cdot \sin x.$$

By symmetry we have $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -\sin x \cdot \sin y$ and by continuity (of the second-order partials) we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \cos x \cdot \cos y$. Therefore

$$\begin{aligned}
f(x_0, y_0) &= \sin \tfrac{\pi}{2} \cdot \sin \tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}, \\
f_x(x_0, y_0) &= \cos \tfrac{\pi}{2} \cdot \sin \tfrac{\pi}{3} = 0, \\
f_y(x_0, y_0) &= \sin \tfrac{\pi}{2} \cdot \cos \tfrac{\pi}{3} = \tfrac{1}{2}, \\
f_{xx}(x_0, y_0) &= f_{yy}(x_0, y_0) = -\sin \tfrac{\pi}{2} \cdot \sin \tfrac{\pi}{3} = -\tfrac{\sqrt{3}}{2}, \\
f_{xy}(x_0, y_0) &= f_{yx}(x_0, y_0) = \cos \tfrac{\pi}{2} \cdot \cos \tfrac{\pi}{3} = 0.
\end{aligned}$$

b) We have

$$A(x,y) + \mathbf{b} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} (x,y) + \left(0, \tfrac{1}{2}\right).$$

Taking the dot product with (x, y) and adding $f(x_0, y_0)$ gives

$$\begin{aligned} Q(x,y) &= (x,y) \bullet A(x,y) + \mathbf{b} \bullet (x,y) + f(x_0,y_0) \\ &= -\frac{\sqrt{3}}{2}x^2 - \frac{\sqrt{3}}{2}y^2 + \frac{1}{2}y + \frac{\sqrt{3}}{2}. \end{aligned}$$

c) The surface $z = Q(x, y)$ is an elliptic paraboloid. Its equation can be rewritten as

$$Z = X^2 + Y^2,$$

where

$$X = x, \qquad Y = y - \frac{1}{2\sqrt{3}}, \qquad Z = -\frac{2}{\sqrt{3}}z + \frac{13}{12}.$$

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Problem 3

A *minimal surface* is a surface with the least surface area among all surfaces with the same boundary. Soap bubbles are a real world example. Minimal surfaces are precisely those satisfying the following property: if a piece of the surface is given by the equation $z = f(x, y)$, where f has continuous second order partial derivatives, then

$$(1 + z_x^2)z_{yy} + (1 + z_y^2)z_{xx} = 2z_x z_y z_{xy}.$$

Let $S \subset \mathbb{R}^3$ be the surface defined by the equation $e^z \sin y = \sin x$.

a) Use implicit differentiation to compute the first and second order partial derivatives of z with respect to x and y.

b) Show that the surface S is minimal.

Solution

a) The surface is given by $e^z \sin y - \sin x = 0$ so differentiating with respect to x and y, while keeping in mind that $z = z(x, y)$, gives

$$0 = z_x e^z \sin y - \cos x,$$
$$0 = z_y e^z \sin y + e^z \cos y.$$

Solving for z_x and z_y gives

$$z_x = \frac{\cos x}{e^z \sin y} = \frac{\cos x}{\sin y} \cdot \frac{\sin y}{\sin x} = \cot x, \qquad z_y = -\frac{\cos y}{\sin y} = -\cot y.$$

Here we used the fact that the identity $e^z \sin y = \sin x$ holds on the surface. Differentiating further gives

$$z_{xx} = \tfrac{\mathrm{d}}{\mathrm{d}x} \cot x = -\frac{1}{\sin^2 x},$$
$$z_{yy} = -\tfrac{\mathrm{d}}{\mathrm{d}y} \cot y = \frac{1}{\sin^2 y},$$
$$z_{xy} = z_{yx} = 0.$$

b) Substituting gives

$$\begin{aligned}(1+z_x^2)z_{yy}+(1+z_y^2)z_{xx} &= \frac{1}{\sin^2 y}(1+\cot^2 x)-\frac{1}{\sin^2 x}(1+\cot^2 y)\\ &= \frac{1}{\sin^2 y}\cdot\frac{1}{\sin^2 x}-\frac{1}{\sin^2 x}\cdot\frac{1}{\sin^2 y}\\ &= 0 = 2z_x z_y z_{xy},\end{aligned}$$

and we are done. Here we used the fact that $z_{xy}=0$ and that for all $\theta \neq k\pi$ the following identity holds:

$$1+\cot^2\theta = \frac{1}{\sin^2\theta}.$$

Note: The problem can also be solved by showing

$$(1+x_y^2)x_{zz}+(1+x_z^2)x_{yy} = 2x_y x_z x_{yz},$$

$$(1+y_x^2)y_{zz}+(1++y_z^2)y_{xx} = 2y_x y_z y_{xz}.$$

That would also cover points with $x=m\pi$ and $y=n\pi$, for which z cannot be written as a function of x and y (recall the Implicit Function Theorem).

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Problem 4

The *body mass index* of a human is defined as

$$b = \frac{m}{h^2},$$

where m is mass and h is height. Suppose that, due to measurement errors, the values $m + \Delta m$ and $h + \Delta h$ are observed instead of the actual values m and h respectively. Further suppose that $|\Delta m|$ is at most 0.1% of m and $|\Delta h|$ is at most 0.1% of h, i.e. the error is at most 0.1% up or down for both values. Using differentials, estimate the resulting error in b (as a percentage of b) in the worst case scenarios.

Solution

The differential of b is

$$\mathrm{d}b = \frac{\partial b}{\partial m}(m,h)\,\mathrm{d}m + \frac{\partial b}{\partial h}(m,h)\,\mathrm{d}h = \frac{1}{h^2}\,\mathrm{d}m - \frac{2m}{h^3}\,\mathrm{d}h.$$

Since $b = b(m,h)$ is differentiable (unless $h = 0$), if $b + \Delta b$ denotes the miscalculated value, we have $\Delta b \approx \mathrm{d}b$ and therefore

$$\frac{\Delta b}{b} \approx \frac{1}{b} \cdot \frac{1}{h^2}\Delta m - \frac{1}{b} \cdot \frac{2m}{h^3}\Delta h.$$

Substituting $b = \dfrac{m}{h^2}$ on the right-hand side gives

$$\frac{\Delta b}{b} \approx \frac{\Delta m}{m} - 2\frac{\Delta h}{h}.$$

Since $\frac{|\Delta m|}{m} \leq 0.1\%$ and $\frac{|\Delta h|}{h} \leq 0.1\%$, this gives

$$\frac{|\Delta b|}{b} \approx 0.3\%$$

in the worst case scenarios (either b is overestimated by overestimating m and underestimating h, or b is underestimated by underestimating m and overestimating h).